Pythagoras and His Theorem

Solution Commentary:

Solution of Main Problems:

1. Because triangles BKC and BCA are similar by AAA, BK/BC = BC/BA. By substitution of given lengths, this becomes x/a = a/c or a^2 = xc. Similarly, because triangles CKA and BCA are similar by AAA, AK/AC = AC/BA. By substitution of given lengths, this becomes (c-x)/b = b/c or b^2 = c(c-x). Thus, a^2 + b^2 = xc + c(c-x) = c^2. Note the geometry involved via the equality of areas involved, as rectangle HIKB with sides x and c has an area equal to that of square BCFG, while rectangle AKIJ with sides (c-x) and c has an area equal to that of square CAED.

2. Since parallelogram BHIK and triangle BHC have the same base BH and same height HI, area(parallelogram BHIK) = 2 x area(triangle BHC). Similarly, since parallelogram GBFC and triangle GBA have the same base GB and same height GF, area(parallelogram GBFC) = 2 x area(triangle GBA). But, triangles BHC and GBA are congruent by SAS (i.e. GB = BC, BA = BH, and angle GBA = right angle GBC + angle ABC = right angle HBA + angle ABC = angle HBC). Thus, area(rectangle BHIK) = area(square GBCF). By drawing segments CJ and EB, a similar argument concludes that area(rectangle IJAK) = area(square AEDC). That is, a^2 = b^2 + c^2.

3. Start with given triangle ABC such that AB^2 = BC^2 + CA^2. Assume that angle BCA is not a right angle. At point C, construct a segment CD perpendicular to segment CA such that CD = CB. Construct segment DA, forming right triangle CDA. But then, DA^2 = DC^2 + CA^2 = BC^2 + CA^2 = AB^2, or DA = AB. Now, triangles CBA and CDA are congruent by SSS, which leads to a contradiction of our assumption that angle BCA was not a right angle. Thus, triangle ABC is a right triangle.

Extension 1: The visual arguments will vary but be something like the following:

- The figures are congruent squares with side length (a+b). The white region on the left is a square of side length c and area c^2 because the two shared angles at each vertex are complementary. The white region on the right involves two squares with areas a^2 and b^2. Thus, a^2 + b^2 = c^2.

- The outside frame forms a square of side length c and area c^2, again because the two shared angles at each vertex are complementary. Also, the smaller white region is a square because of the perpendicular intersections occurring at each of its vertices. Thus, the overall area equals the sum of the four interior areas, or c^2 = 4[½(ab)] + (a-b)^2 = 2ab + a^2 - 2ab + b^2 = a^2 + b^2.

- A square is constructed on each side of the initial triangle, forming areas a^2, b^2, and c^2. Then, as if moving pieces of a puzzle, the sub-regions of the square a^2 and the square b^2 can be arranged to form the square c^2. It becomes a great exercise in geometry to show that all of the pieces fit
properly. This proof can easily be connected to the previous “Behold!” proof, if two additional dotted lines are constructed:

Extension 2: The three regions are two congruent right triangles and an isosceles right triangle, again because the two shared angles at the lower vertex are complementary. Then, the combined area of these three regions is $2[\frac{1}{2}(ab)] + \frac{1}{2}c^2 = ab + \frac{1}{2}c^2$. However, when the three regions are viewed as a trapezoid, the combined area is $\frac{1}{2}(b+a)(b+a) = \frac{1}{2}a^2 + ab + \frac{1}{2}b^2$. Thus, since the two expressions have to be equal (i.e. same area is involved), we have that $\frac{1}{2}a^2 + \frac{1}{2}b^2 = \frac{1}{2}c^2$ or $a^2 + b^2 = c^2$.

Open-Ended Exploration: This exploration is an opportunity for students to rethink through the mathematical tools they have acquired. Plus, it is an opportunity to share Loomis (1968) and its collection of 367 proofs of the Pythagorean Theorem. In his Foreward, Loomis claims that there are only “four kinds” of proofs, then asserts that the number of algebraic and geometrical proofs is limitless. On his web site http://www.cut-the-knot.org/pythagoras/index.shtml, Alexander Bogomolny points out some known errors in Loomis’ collection proofs, while also providing some great additional information about the Pythagorean Theorem, including 81 different proofs. Finally, Maor (2007) offers his candidate for the “most unusual” proof, which involves the squaring of infinite series and is found in an early 1900’s calculus text by Landau.

Teacher Commentary:

When exploring Problems #1 and #2, make sure students both see and understand the differences between the two proofs. As Heath (1956, pp. 349-356) points out, these differences reflect both the nature of assumptions and style, yet both involve the Greek idea of “applying areas.”

Point out that the Converse of the Pythagorean Theorem suggests a useful process for classifying a triangle as being right, obtuse, or acute. If $c$ is the longest of the three sides, then:

- If $a^2 + b^2 = c^2$, the triangle is right.
- If $a^2 + b^2 > c^2$, the triangle is acute.
- If $a^2 + b^2 < c^2$, the triangle is obtuse.
Ask students if the sorting process is obvious or is a proof necessary? Also, ask students about the case where the there are two “longest” sides of length c.

When discussing the visual “Behold!” proofs, raise issues. For example, in the second “Behold!” proof, is it possible to know the desired relationship without doing a significant amount of algebra. Actually, the algebra can be avoided by a translation argument, as shown dynamically at http://en.wikipedia.org/wiki/File:Pythagoras-2a.gif. Also, ask students to investigate a “faulty” visual proof, as provided and corrected by Alexander Bogomolny at http://www.cut-the-knot.org/pythagoras/FaultyPythPWW.shtml.

Bogomolny also mentions Professor Raymond Smullyan's book 5000 B.C. and Other Philosophical Fantasies (1984), which describes an experiment Smullyan did a geometry class. First, he drew a right triangle with squares on each side, noting that the hypotenuse’s square was certainly larger than the other two squares. His question to the class: “Suppose these three squares were made of beaten gold, and you were offered either the one large square or the two small squares. Which would you choose?” The result was a 50-50 split in the class, with both groups quite surprised when Smullyan proved that the two choices were equal.

As a good source of writing projects, students can explore any of the following ideas relative to Pythagoras and proofs of his famous theorem:

- The naming of eponymic theorems often is suspect. For example, Pythagoras is given full credit name-wise for this theorem, yet Swetz and Kao (1977) and others have documented that the theorem was known prior to Pythagoras’ time. Investigate the misnaming of other eponymic theorems, and try to determine authenticity in each case.
- Some historians claim the Babylonians knew aspects of the Pythagorean Theorem. For example, one tablet from Old Babylonia (ca. 1900 B.C.) read: 4 is the length and 5 the diagonal. What is the breadth? Its size is not known. 4 times 4 is 16. 5 times 5 is 25. You take 16 from 25 and there remains 9. What times what shall I take in order to get 9? 3 times 3 is 9. 3 is the breadth. Do you think such a problem is ample evidence that the Babylonians knew the special properties of a right triangle viz. the Pythagorean Theorem? Investigate other findings, as well as evidence that the Babylonians built tables of Pythagorean triples. Some resources are Maor (2007), Friberg (2005), and http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Babylonian_Pythagoras.html.
- Investigate other examples of U.S. Presidents exploring mathematical ideas. A good start is Dunham’s (1994) chapter “Hypotenuse.” Other resources are Wielenberg (1990), Smith (1932), Frisinger (1976), and Bompant (1994).
- In the film The Wizard of Oz (1939), the Scarecrow receives his requested brains from the wizard and then recites a “mangled” version of the Pythagorean Theorem: The sum of the square roots of any two sides of an isosceles triangle is equal to the square root of the remaining side. Explore this statement. Is there a geometrical system where it would be correct? Some
helpful resources are Leake (1986), Pancari & Pace (1987),
http://mathbits.com/MathBits/MathMovies/OzMath.pdf, or

- In his Ascent of Man (1976, p. 160), Bronowski makes this bold claim: To this
day, the theorem of Pythagoras remains the most important single theorem in
the whole of mathematics. Do you agree or disagree with this claim? On what
criteria can one make decisions such as “most important”?
- Investigate the connections between the Pythagorean Theorem and the
discovery of irrational numbers. For example, if the essence of the
Pythagorean Theorem was known prior to the time of Pythagoras, why weren’t
irrational numbers discovered earlier as well? Maor (2007), Van der Waerden,
(1988), Gow (1968), and Friberg 200&) are some good initial resources.

Additional References:

Teacher. February, pp. 23-32.


Mathematics. World Scientific.

World Scientific.

April, pp. 301-307.

Reprint.

Publications.

Leake, L. (1986). “Did the scarecrow really get a brain? An analysis of the
Scarecrow’s Pythagoras-like statement in The Wizard of Oz.” Mathematics Teacher,

Teachers of Mathematics.

University Press.


