Thales and His Semicircle Theorem

Solution Commentary:

Solution of Main Problems:

1. Begin by constructing segment DO. Then, by the definition of a circle, \( AO = DO = BO \), and triangles AOD and DOB are isosceles. By Thales’ third theorem, this implies that angles ODA and OAD are congruent. Similarly, angles ODB and OBD are congruent.

   \[ \text{Diagram} \]

   The next step in a proof depend on what assumptions can be made or what else has been proven:
   - If we know that a triangle’s angle sum is two right angles, we have angle ADB equal to the sum of angles ODA and ODB and equal to a right angle.
   - Or, if we know about the exterior angle theorem, we can form ray AD and find point E on the ray and exterior to the circle. Then, angle EDB is an exterior angle equal to the sum of angles DAB and DBA. But, this implies that angles EDB and ADB are congruent and form a straight angle, making angle ADB a right angle.

   Note: If you already know the relationship that the measure of angle inscribed in a circle is half the measure of the central angle with the same intercepted arc, then angle ADB is half the measure of straight angle AOB, or a right angle.

2. Several proofs are possibly:
   - **Proof A:** Start with triangle ABC with right angle BAC. Construct line \( p \) parallel to AB passing through point C, and a line \( q \) parallel to AC passing through point B. Lines \( p \) and \( q \) will intersect at point D. Quadrilateral BACD is a parallelogram (by construction) and a rectangle (adjacent angles supplementary and thus are all right angles). Construct diagonals AD, intersecting hypotenuse BC (also a diagonal) at point E. By properties of a rectangle, \( AE = CE = DE = BE \), making point E the center of a circumcircle \( O \) for triangle ABC. And, point E on the hypotenuse BC implies that BC is a diameter of the circumcircle \( O \).
Note: Can students prove that this circumcircle O is unique?

- **Proof B:** Assume that the converse is not true. That is, given right triangle ABC, the center E of the circumcircle O is not on hypotenuse BC. Construct a diameter through point B and center E, intersecting the circumcircle O at point D. Construct segment AD. Then, by Thales’ fourth theorem, angle BAD is a right angle, which leads to a contradiction of the fact that a perpendicular to line AB at point A is unique. Thus, the converse must be true.

Note: Can students prove that the perpendicular is unique?

- **Proof C:** Use linear algebra. Given right triangle ABC, bisect the hypotenuse BC to find point M. Draw circle O using center M and radius MC. Viewing M as the origin, \( \overrightarrow{MB} = -\overrightarrow{MC} \). Angle BAC a right angle implies that \( \overrightarrow{AB} \cdot \overrightarrow{AC} = 0 \). By substitution, \( (\overrightarrow{MB} - \overrightarrow{MA}) \cdot (\overrightarrow{MA} - \overrightarrow{MC}) = \overrightarrow{MB} \cdot \overrightarrow{MA} + \overrightarrow{MC} \cdot (\overrightarrow{MA} + \overrightarrow{MB}) \). By properties of a vector dot product, \( \overrightarrow{MB} = 0 \) or \( \overrightarrow{MB} = \overrightarrow{MA} \) and point A is also on circle O, making it the circumcircle for right triangle ABC.

**Extension 1:** Van der Waerden (1988, pp. 87-88) is the prime advocate of this claim. The diagram is as follows:

By construction, AC = CD and angles BAC and CDE are congruent right angles. Also, angles BCA and ECD are congruent vertical angles. Thus, triangles ACB and
DCE are congruent by ASA, making DE = AB. Given that the proof involves two of Thales four theorems, it is likely that Thales could know and use this method. In real life, the method has problems, varying from an inability to turn exactly 90 degrees, walk straight lines, and be accurate in determining collinearity, plus the ship will probably move during the implementation of the method.

**Extension 2:** Allman (1889, p. 9) provides this quote from Plutarch. The diagram is as follows, assuming the sun's rays are parallel:

![Diagram](https://via.placeholder.com/150)

Triangles CDE and FEG are similar since angles CDE and FEG are congruent right angles and angles CED and FGE are congruent corresponding angles on parallel lines. Since the corresponding sides of similar triangles are proportional, CD/FE = DE/EG. Now, DB was a known measurement of the pyramid, while the lengths of BE, EG, and FE could be measured. Thus, CD = [(FE)(DB+BE)]/(EG). Reconsidering Thales five theorems, none of them relate to similar triangles and the proportionality of their corresponding sides, an idea which seems to first appear in Euclid's *Elements*. Yet, this relationship is now known as Thales Theorem as well (see Patsopoulos and Patronis, 2006).

**Open-Ended Exploration:** Students should discover that it is difficult to prove geometrically such a simple statement. Historians provide little insight. For example, Gow (1968, p. 141) suggest that Thales did not prove it but "relied on intuition, or as the Eudemian summary has it, attacked the question empirically." Ask students to decide if that is equivalent to cutting a circle out of paper and folding it across its diameter, basically "showing" the theorem's truth by both suggested meanings? Gow later adds that Euclid did not think the theorem "worthy" of proof, in that it could "be inferred from definitions 17 and 18 to Book I." As if your students agree?

**Teacher Commentary:**

Emphasize that these explorations of the theorems are speculations, as Thales’ actual proofs are unknown. Also, students should be aware that multiple proofs are known for justifying Thales’ fourth theorem, including trigonometry and vector geometry. Also, this Thales Theorem is found in Euclid’s *Elements* as Proposition 31 (Book III).
Prathap (1996) provides additional information about Thales' establishment of the process of demonstrative geometry, focusing on the shift of scientific thought from *mythos* (i.e. understanding the world via traditional stories) to *logos* (i.e. understanding the world via reasoning). He cites Greek historian Kitto, who described Thales' work as: "It is as if the human mind for the first time took its toes off the bottom and began to swim, and to swim with astonishing confidence."

In problem #2, students should consider and evaluate the different proofs that are possible. The differences are tied to the mathematics that is being assumed, plus issues of uniqueness need to be raised. Other proofs are possible as well, such as using analytic geometry (e.g. distances, slopes) or trigonometry.

In Extension Problems 1 and 2, encourage students to enact the methods in real situations, such as finding the distance of an object from their position or the height of a building. The students should discover that the methods provide rough estimates, but both could admit error in multiple ways.

As a good source of writing projects, students can explore any of the following ideas relative to Thales and his geometrical ideas:

- The naming of eponymic theorems often is suspect. For example, Van der Waerden (1988, p. 88) writes: 
  
  Pamphile, as reported by Diogenes Laërtius, says that Thales was the first to construct a circle about a right triangle and that, in honor of this discovery, he sacrificed a bull. Hence, the proposition that an angle inscribed in a semicircle is a right angle, is ascribed to Thales. On the other hand, this proposition is related to certain calculations concerning chords and their apothems, which occur in Babylonian mathematics.
  
  Yet, Heath (1956, p. 319), in his translation of Euclid’s *Elements*, suggests that the same story about sacrificing the ox is told regarding the discovery of the Pythagorean Theorem (Proposition 47, Book 1). Investigate the misnaming of other eponymic theorems, and try to determine authenticity in each case.

- Investigate the “second” eponymic Thales Theorem regarding the claim that the corresponding sides of similar triangles are proportional. Prove both this claim and its converse. Is the mathematics required something that Thales would have access to?

- Some claim that Thales' knowledge of the veracity of his fourth theorem, whether by proof or empirical reasoning, suggests that he also knew that the sum of the angles in a triangle is equal to two right angles. That is, connecting the right angle’s vertex to the circle’s center creates two isosceles triangles with pairs of congruent base angles $\alpha$ and $\beta$. Thus, $\alpha + \beta$ equals the right angle, or the sum of the triangle’s angles is $2(\alpha + \beta)$ or two right angles. Yet, Allman (1889, p. 12) suggests that Thales already knew this because of his contemplation of Egyptian tiled floors (e.g. both six equilateral triangles and four square tiles meet around a single vertex, implying that three equilateral triangle angles equal two right angles). Investigate and document the history.
of claims that a triangle has an angle sum of two right angles, leading into non-Euclidean disruptions of this claim.

- Thales is viewed as the leader of the Iconic School, which is credited with the early foundations of geometry prior to Pythagoras. Investigate the mathematical contributions of other members of this Ionic School, many of them students of Thales. Examples are Mamercus, Anaximander of Miletus, Anaximenes, Anaxagoras of Clazomenae, and Enopides of Chios.

**Additional References:**


