Archimedes and His Mechanical Method

Solution Commentary:

Solution of Main Problems:

1. Due to isosceles triangles AOC and AOD, angles CAO and DAO are 45° degrees. Then, right angles ABE and ABF imply that angles AEB and AFB are 45° as well. Since triangles ABE and ABF are isosceles right triangles, AB = BE = BF = 2r.

2. First, note that ZT = x as well, because of isosceles right triangle AZT. From right triangle AZR, \( x^2 + y^2 = (AR)^2 \). Since triangles AZR and ARB are similar by AAA, \( \frac{AR}{AZ} = \frac{AB}{AR} \) or \( (AR)^2 = (AZ)(AB) = (x)(2r) \). By the transitive property, \( x^2 + y^2 = (x)(2r) \).

3. Geometrically,
   - The rotation of triangle EAF generates a large cone
   - The rotation of the circle with center O generates a sphere
   - The rotation of rectangle EFGH generates a cylinder
   - The rotation of segment PQ generates a circle
   - The rotation of triangle CAD generates a small cone
   - The rotation of arc CAD generates a hemisphere
   - \( \pi x^2 \) represents the area of the circle generated by segment ZT, and is a “slice” of cone generated by rotating triangle EAF
   - \( \pi y^2 \) represents the area of the circle generated by segment ZR, and is a “slice” of sphere generated by rotating the circle with center O
   - \( \pi (2r)^2 \) represents the area of the circle generated by segment ZP, and is a “slice” of the cylinder generated by rotating rectangle EFGH

4. First, substituting lengths, \( \frac{\text{sphere+cone}}{\text{cylinder}} = \frac{AO}{AW} = \frac{r}{2r} = \frac{1}{2} \), which implies that
   \[ 2(\text{spheres}) + 2(\text{cones}) = 1(\text{cylinder}) \]

5. By substitution, \( 2(\text{spheres}) + 2(\text{cones}) = 1(\text{cylinder}) = 3(\text{cones}) \), or \( 2(\text{spheres}) = 1(\text{cones}) \).

6. View the small cone as a transformation of the large cone using a scale parameter of \( AO/AB = \frac{1}{2} \). Then, the volume is scaled by a factor of \( \left( \frac{1}{2} \right)^3 = \frac{1}{8} \)
   and by substitution, \( 2(\text{spheres}) = 1(\text{large cone}) = 8(\text{small cones}) \), or \( 1(\text{sphere}) = 4(\text{small cones}) \). But, this implies that \( 1(\text{hemisphere}) = 2(\text{cones}) \), which is what was to be proven.

Extension 1: Inscribe similar regular n-gons (e.g. 3-gon or equilateral triangles) in circles C1 and C2. Bisect all of the polygon’s sides, construct perpendiculars at the midpoints that will intersect the circle to provide new vertices for building a regular 2n-gon:
The necessary area relationship will be preserved in both circles: (area n-gon) < (area 2n-gon) < (area circle). Continue this process in both circles \( C_1 \) and \( C_2 \) until you construct large enough similar regular polygons \( P_1 \) and \( P_2 \) such that (area \( R \)) < (area \( P_2 \)) < (area circle \( C_2 \)). By Eudoxus’ assumed “known fact”, we know \( \frac{P_1}{P_2} = \frac{(d_1)^2}{(d_2)^2} \), which by our assumption, also implies that \( \frac{P_1}{P_2} = \frac{A_1}{S} \). But, the latter relationship is a contradiction since \( A_1 > P_1 \) and \( S < P_2 \).

**Extension 2:** Similar to the process used for the triangle, a sphere can be subdivided in a large number of very small circles tightly-packed, which when connected to the sphere’s center will form a large number of cones with height equal to the radius \( r \) and whose combined volumes. The key idea is that the combined area of the bases of these “radial” cones will equal the surface of the sphere, while their combined volume will equal the volume of the sphere. Three sample cones are shown below:

Thus, (Volume of radial cones with height \( r \)) = (Volume of sphere) = (Volume of single cone with height \( r \) and a base whose area is equal to the surface area of the sphere). But, because Archimedes showed that (Volume of hemisphere) = (Volume of 2 cones with height \( r \) and base equal to a great circle), we have (Volume of sphere) = (Volume of 4 cones with height \( r \) and base equal to a great circle) = (Volume of 1 cone of height \( r \) and base equal to four great circles). Thus, combining the two statements, we can conclude that the surface area of the sphere is equal to the area of four great circles (i.e. surface area sphere = \( 4 \pi r^2 \)).
Open-Ended Exploration: This is a great opportunity for students to see the power of integration in calculating both volumes of revolution and surface area, thereby allowing direct comparison to the methods used by Archimedes.

Teacher Commentary:

Explorations of these problems is enhanced by showing students a video about Archimedes lost “method” and the re-discovery of a palimpsest. Two suggested videos are:


The first is my preference, but either will suffice. Plus, the second includes printable materials for teachers.

Many other videos (e.g. Google Tech Talks) and discussions regarding Archimedes palimpsest are available on the web, but are of varied quality. As the number and links keep changing, a current search and preview on your part is best. Nonetheless, the web site for the Archimedes Palimpsest Project is of high quality and is strongly recommended (http://www.archimedespalimpsest.org/palimpsest_making1.html).

This web site is complemented by the texts by Stein (1999) and Netz & Noel (2007).

To enhance these explorations, show students physical models of cones, hemisphere, and cylinder that fit Archimedes’ criteria. These models are available on line (e.g. at NASCO). Students are challenged when shown a large cone and asked the relative size of the “small” cone formed if the large cone was cut into two pieces by a plane perpendicular to the height at at the height’s midpoint. Unfortunately, showing a lack of understanding of scaling principles, the students are surprised by the 1/8 relationship. Finally, you can even hang the respective pieces to illustrate the Law of the Lever.

Observant students will see the process of “exhaustion” being used to approximate the modern use of limits. Some may conclude that Archimedes’ “fudging” is unwarranted (e.g. circular bases of cones covering surface of sphere in Extension Problem #2), but such provides a great opportunity to discuss the relative roles and powers of limits.

When exploring the proof of Eudoxus’s Theorem in Extension Problem #1, stress that the “known fact” is a variation of Euclid’s Proposition 19 (Book 6): Similar shapes are to one another in the duplicate ratio of their corresponding sides. Here, Euclid is using the term “duplicate ratio” to mean “ratio of their squares.”

As a good source of writing projects, students can explore any of the following ideas relative to Archimedes and his Method:
In Archimedes’ proof relating the volume of the hemisphere and the inscribed cone, Archimedes uses Democritus’ relationship that \( \frac{1}{3} \) (volume of cone inscribed in a cylinder) = \( \frac{1}{3} \) (volume of cylinder). Unfortunately, Democritus only suggested this relationship and was not able to prove it. To his rescue, Eudoxus (370 B.C.) provided the proof, as documented by Archimedes and as found in Euclid’s Proposition. Investigate and determine the approach of this proof. Does it also use the process of “exhaustion”? Some useful resources are Allman (1889), Gow (1968), Heath (1963), and Van der Waerden, B. (1988).

Eudoxus’ reasoning in Extension Problem #1 illustrates the early uses of proof by contradiction. Other examples are Euclid’s proof that the number of primes is infinite and the irrationality of \( \sqrt{2} \). Investigate and document the history of different proof techniques, especially early uses of proof by contradiction. Useful references are http://www.math.wustl.edu/~sk/eolss.pdf and http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Eudoxus.html.

Eudoxus’ reasoning in Extension Problem #1 illustrates the early uses of proof by exhaustion. Investigate and document the history of different proof techniques, especially early uses of proof by exhaustion, connecting it to the co-development of limit concepts. Some useful resources are Allman (1889), Gow (1968), Heath (1963), and Van der Waerden, B. (1988).

In Extension Problem #2, Archimedes’s statement is considered by some to be “one of the finest examples of bold analogy in the history of mathematics.” Investigate and document the history of other good examples of the use of analogy in mathematics. Some resources are Polya (1968), Corfield (2003), and Ippoliti (2008).

The value of \( \pi \) was used to enhance Archimedes’ proof. Investigate the history underlying the appearance and acceptance of the value \( \pi \), plus the eventual development of our formula for area of circle. Two resources are Beckmann (1976) and http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Pi_through_the_ages.html.

Petr Beckmann, a Czechoslovakian scientist, interpreted history as a continual combat between two societal groups, the thinkers and the thugs. He even formulated Beckmann’s Law: “In the conflict between the thinkers and the thugs, the thugs always win, but the thinkers always outlive them.” One example of this Law is Archimedes (i.e. the Greeks) and Marcellus (i.e. the Romans), Investigate and document other examples specific to mathematics and its history. (Eves, 1983)

Additional References:


