Archimedes’ Estimation of Pi

Historical Context:
- When: 287 - 212 B.C.
- Where: Syracuse, Sicily (Greece)
- Who: Archimedes
- Mathematics focus: Investigation of the calculation of a value of pi.

Suggested Readings:
- Archimedes and his contributions to mathematics and science:  
  http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Archimedes.html
- Basic information about pi:  
  http://mathworld.wolfram.com/Pi.html
- NCTM’s Historical Topics for the Mathematics Classroom (1969): “The number π” (pp. 148-153)
- Key search words/phrases: Archimedes, Greek geometry, pi, approximation

Problem to Explore:
Investigate the use of the method of exhaustion to approximate a value of pi, sometimes known as Archimedes’ constant..

Why This Problem is Important:
- Perhaps is the first theoretical calculation of an approximate value for pi.
- Provides insight into Archimedes’ use of the method of exhaustion for investigating mathematical relationships, leading to the idea of a limit.

Problem Solving Experiences:
Prior to Archimedes, values of pi were known and used, but usually were estimates based on measurements. The known “fact” underlying these estimates was that the ratio of a circle’s circumference to its diameter is a constant value. The history of this “fact” is not known, except that it probably preceded the times of the Egyptians and Babylonians.
Archimedes produced perhaps the first theoretical calculation of pi, using the same “fact.” In his text *Measurement of a Circle*, Archimedes offered the result of his calculations as Proposition 3: *The ratio of the circumference of any circle to its diameter is less than $3\frac{1}{7}$ but greater than $3\frac{10}{71}$.

To understand Archimedes’ argument, consider these construction steps:

- Draw a semicircle AXB with center O and diameter AB.
- Construct tangent line CA to semicircle at point A and such that angle AOC is one-third of a right angle. (Note: The angle can be created by first constructing an equilateral triangle on segment AO using Euclid’s first proposition.)
- Construct point D on AC such that segment OD bisects angle AOC. Repeat this process with point E, F, G on AC such that segment OE bisects angle AOD, segment OF bisects angle AOE, and segment OG bisects angle AOF.
- Construct point H on ray GA such that GA = AH.

1. Angle GOH is what fraction of a right angle? If this process was repeated around the circumference of the entire circle, Segment GH would be the side of a circumscribed regular polygon with how many sides?

Archimedes then used three ideas without justification. The first idea essentially was Euclid’s Proposition 3 (Book VI): *If an angle of a triangle is bisected by a straight line cutting the base, then the segments of the base have the same ratio as the remaining sides of the triangle; and, if segments of the base have the same ratio as the remaining sides of the triangle, then the straight line joining the vertex to the point of section bisects the angle of the triangle.* The second idea was that $\frac{\sqrt{3}}{1} > \frac{265}{153}$. The third idea was that for a 30°-60°-90° triangle, the Pythagorean relationship implies that the side lengths are multiples of $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, and 1.
2. By focusing on OD as the bisector of angle COA in triangle COA and using Archimedes’ three “unjustified” ideas, show that \( \frac{OA}{AD} > \frac{571}{153} \).

3. Using the Pythagorean Theorem and the results in Problem #2, show that
\[
\frac{OD^2}{AD^2} > \frac{571^2 + 153^2}{153^2} = \frac{349450}{23409},
\]
from which Archimedes claims \( \frac{OD}{AD} > \frac{591/8}{153} \).

Having focused on segment OD as the bisector of angle COA in triangle COA, Archimedes then repeats his reasoning process:

- By focusing on segment OE as the bisector of angle COD in triangle COD, he repeats the same calculation process to show that
\[
\frac{OA}{AE} = \frac{DO + OA}{DA} > \frac{591/8 + 571}{153} = \frac{1162/8}{153},
\]
and
\[
\frac{OE^2}{AE^2} > \frac{(1162/8)^2 + 153^2}{153^2},
\]
which implies that \( \frac{OE}{AE} > \frac{1172/8}{153} \).

- By focusing on segment OF as the bisector of angle COE in triangle COE, he repeats the same calculation process to show that
\[
\frac{OA}{AF} > \frac{1172/8 + 1162/8}{153} = \frac{2334/4}{153},
\]
which implies that \( \frac{OF}{AF} > \frac{2339/4}{153} \).

- And, by focusing on segment OG as the bisector of angle COF in triangle COF, he repeats the same calculation process to show that
\[
\frac{OA}{AG} > \frac{2339/4 + 2334/4}{153} = \frac{4673/2}{153}.
\]

From this information, Archimedes is able to place a bound on the circumference of the circle and indirectly obtain a value of pi, since he knew that the circumference of the circle was less than the perimeter of the regular polygon with 96 sides.

4. Show that the perimeter of the regular polygon with 96 sides is less than \( 3\frac{1}{7} \) of the circle’s diameter AB, or that \( \pi < 3\frac{1}{7} \).

Archimedes then shifted his focus to providing a lower bound by basically inscribing a 96-sided regular polygon. His initial construction of the side of the 96-gon was as follows:
Draw a semicircle AXB with center O and diameter AB.

With point C on the circle, construct chord CA such that angle CAB is one-third of a right angle.

With point D on the circle, construct chord AD as bisecting angle BAC. Repeat this process with point E, F, G on the circle such that chord AE bisects angle BAD, chord AF bisects angle BAE, and chord AG bisects angle BAF.

Now, the chord GB is then the side of a regular 96-gon inscribed in the circle. Without exploring all of Archimedes' reasoning for this case, it is important to note two key aspects:

- Each triangle BAC, BAD, BAE, BAF, and BAG are right triangles because they involve an angle inscribed in a semicircle.
- Triangles BAC, YAC, and BYD are similar, which creates the necessary ratios of side lengths.

Again, Archimedes performs a repetitive process, eventually achieving the ratio

\[
\frac{BG}{AB} > \frac{66}{2017\frac{1}{4}}
\]

and then shows that

\[
\text{Perimeter of 96-gon} < \left(\frac{96(66)}{2017\frac{1}{4}}\right) \cdot (AB) < \left(\frac{3}{71}\right)AB,
\]

implying the circle's circumference is greater than \(3\frac{10}{71}\) times the circle's diameter.

Thus, by both inscribing and circumscribing regular 96-gons, Archimedes has shown that

\[
3\frac{10}{71} < \frac{\text{circumference of circle}}{\text{diameter of circle}} < 3\frac{1}{7}, \quad \text{or} \quad \frac{3}{71} < \pi < \frac{3}{7}
\]

because by definition, \(\pi\) is the ratio of a circle's circumference to its diameter. That is, Archimedes has shown that \(3.140845070... < \pi < 3.142857142...\). In fact, by taking the average of Archimedes' two bounds, the estimated value of \(\pi\) is 3.141851107..., accurate to 3-decimal places. And, Archimedes' estimate bounds could be improved further by continuing his process using regular 192-gons, 384-gons, etc. In fact, this is exactly what was done by several Chinese mathematicians. In the 3\textsuperscript{rd}-century, Liu Hui inscribed a 3072-gon in a circle to produce a value of \(\pi \approx 3.14159\), followed in the
5th-century by Tsu Chung Chi, who used an inscribed 24576-gon to produce a value of \( \pi \approx 3.1415929 \).

**Extension and Reflection Questions:**

**Extension 1:** Evidence of approximations of values for \( \pi \) exists as early as 2000 B.C. These approximations of \( \pi \), defined the “constant” ratio of a circle’s circumference to its diameter, probably were determined by empirical measurements. For each of the following historical situations, determine the approximate decimal value of \( \pi \).

- **About 2000 BC**, the Babylonians claimed a circle’s area equals 0;5 (i.e. \( \frac{1}{12} \)) times the square of the circle’s circumference. What is the implied value of \( \pi \)?
- **Discovered in excavations near Babylon in 1936**, the Susa tablets claimed a more accurate calculation of a circle’s area occurred is the previous 0;5 was multiplied by 0;57,36. The tablets also stated that the ratio of the perimeter of a regular hexagon to the circumference of its circumscribing circle is 0;57,36. What are these two “improved” values of \( \pi \)?
- **The Rhind Papyrus** provided both an approximate value and some reasoning:
  1. Problem 50 states: “Given a round field with a diameter of 9 khet, what is its area? Take away 1/9 of its diameter, namely 1, leaving a remainder is 8. Multiply 8 times 8, making an area of 64 setjat.” What is the implied value of \( \pi \)?
  2. Problem 48 offers a partial rationale. Trisect the four sides of a square circumscribing a circle of diameter 9. Connect the adjacent “tripoints” to form a non-regular hexagon, whose area approximates the circle’s area. What is the implied value of \( \pi \)?
- **From the Bible**, consider verse 23 of Kings I, chapter 7: “And he made a molten sea, ten cubits from the one brim to the other: it was round all about, and his height was five cubits: and a line of thirty cubits did compass it round about.” What is the implied value of \( \pi \)?
- **In Proposition 2 of Measurement of a Circle**, Archimedes “supposedly” claims: “The area of a circle is to the square on its diameter as 11 to 14.” What is the value of \( \pi \)? Also, what common fraction is equivalent to this decimal?
- **In Problem 31 (Chapter 1) of Jiuzhang Suanshu** (circa 220 BC), the Chinese mathematician Liu Hui equated a circle’s area with the product of half the circle’s circumference multiplied by half its diameter. What is his value of \( \pi \)?

**Extension 2:** Archimedes approach can be replicated using the modern power of trigonometry and decimal notation. Consider these general construction steps:

- Starting with a unit circle, inscribe and circumscribe an equilateral triangle and then bisect each side to inscribe and circumscribe a regular hexagon.
• With the two hexagons or 6-gons, confirm \( OA = 1, \angle AOB = \frac{\pi}{3}, \angle AOD = \frac{\pi}{6}, \angle ODA = \frac{\pi}{2}, \angle OAC = \frac{\pi}{2}, \) \( AD = \sin \left( \frac{\pi}{6} \right), \) and \( AC = \tan \left( \frac{\pi}{6} \right). \)

• Thus, the perimeter of the inscribed hexagon is \( 12(AD) = 12 \sin \left( \frac{\pi}{6} \right) = 6 \) and the perimeter of the circumscribed hexagon is \( 12(AC) = 12 \tan \left( \frac{\pi}{6} \right) \approx 6.92820323, \) producing the relationship \( 6 < 2\pi < 6.92820323 \) or \( 3 < \pi < 3.46410161. \)

• By continually bisecting the sides and producing new inscribed and circumscribed regular polygons, this process can be generalized. In the \( n^{th} \) case, we will have regular \( N \)-gons where \( N = 3(2^{n-1}) \), \( OA = 1, \angle AOD = \frac{\pi}{N}, \angle ODA = \frac{\pi}{2}, \angle OAC = \frac{\pi}{2}, \) \( AD = \sin \left( \frac{\pi}{N} \right), \) \( AC = \tan \left( \frac{\pi}{N} \right), \) the inscribed perimeter \( p_n = 2N \sin \left( \frac{\pi}{N} \right), \) and the circumscribed perimeter \( P_n = 2N \tan \left( \frac{\pi}{N} \right). \)

Set up an Excel chart for \( n = 1, 2, \ldots, 15 \) to show the “limiting” nature of \( \frac{P_n}{2} < \pi < \frac{p_n}{2}. \) Does it appear that \( \lim_{N \to \infty} \frac{P_n}{2} = \lim_{N \to \infty} \frac{p_n}{2} = \pi? \) Can you prove it?

Open-ended Exploration: Assume that \( \pi \) is defined as the “constant” ratio of a circle’s circumference to its diameter. Explore what happens to the value of \( \pi \) in diverse geometrical situations, such as hyperbolic geometry or taxicab geometry.