Archimedes’ Estimation of Pi

Solution Commentary:

Solution of Main Problems:

1. \( \angle \text{GOH} = \angle \text{AOE} = \frac{1}{2} \angle \text{AOF} = \frac{1}{4} \angle \text{AOD} = \frac{1}{8} \angle \text{AOC} = \frac{1}{24} \) right angle. Thus, repeating this in all four quadrants, segment GH is part of a circumscribed regular 96-sided polygon.

2. By Euclid’s Proposition, \( \frac{CO}{OA} = \frac{CD}{AD} \), which implies \( \frac{CO + OA}{OA} = \frac{CD + AD}{AD} \) or \( \frac{CO + OA}{OA} = \frac{CD}{AD} \). But, by Archimedes other “ideas,” this implies

\[
\frac{OA}{AD} = \frac{CO + OA}{CA} = \frac{CO}{CA} + \frac{OA}{CA} = 2 + \sqrt{3} \frac{1}{1} > \frac{306}{153} + \frac{265}{153} = \frac{571}{153}.
\]

3. In right triangle AOD, \( DO^2 = OA^2 + AD^2 \), which implies that \( \frac{OD^2}{AD^2} = \frac{OA^2}{AD^2} + 1 \). By the results of Problem #2, we then have

\[
\frac{OD^2}{AD^2} > \frac{571^2}{153^2} + 1 = \frac{571^2 + 153^2}{153^2} = \frac{349450}{23409}.
\]

We gain the last step of \( \frac{OD}{AD} > \frac{591}{8} \frac{1}{153} \) because \( (591 \frac{1}{8})^2 < 349450 \).

4. Recall that \( AB = 2(OA) \) and \( GH = 2(AG) \), where \( AB \) is the circle’s diameter and \( GH \) is a side of the 96-sided regular circumscribing polygon. Thus,

\[
\frac{AB}{\text{Perimeter of 96-gon}} = \frac{2(OA)}{96[2(AG)]} = \frac{OA}{96(AG)} = \frac{4673 \frac{1}{2}}{(96)(153)} = \frac{4673 \frac{1}{2}}{14688}
\]

or

\[
\text{Perimeter of 96-gon} < \left( \frac{14688}{4673 \frac{1}{2}} \right) (AB). \text{ But, } \frac{14688}{4673 \frac{1}{2}} = 3 + \frac{667 \frac{1}{2}}{2} < 3 + \frac{667 \frac{1}{2}}{2} = 3 \frac{1}{7}.
\]

Finally, since the circle’s circumference is less than the perimeter of the 96-gon, we have that the circle’s circumference is less than \( 3 \frac{1}{7} \) times the circle’s diameter. But, if we knew that the circle’s circumference equaled \( \pi \) times the circle’s diameter, Archimedes has shown that \( \pi < 3 \frac{1}{7} \).

Extension 1:
• Assume \( \left( \frac{1}{12} \right) (2\pi)^2 = \pi^2 \), which implies that \( \frac{1}{3} \pi^2 r^2 = \pi^2 \) or \( \pi = 3 \).

• First, \( 0.57,36 = \frac{57}{60} + \frac{36}{3600} = \frac{96}{100} \), implying situation is \( \left( \frac{96}{100} \right) \left( \frac{1}{12} \right) (2\pi)^2 = \pi^2 \) ith improved value of \( \pi = \left( \frac{100}{96} \right) (3) = 3.125 \). Second, the length of the side of a regular hexagon inscribed inside a circle is equal to the length of the circle’s radius. Thus, \( \frac{6r}{2\pi} = \frac{96}{100} \) or again \( \pi = \left( \frac{100}{96} \right) (3) = 3.125 \).

• First, by Problem 50, the claim is that \( 8^2 = \pi \left( \frac{9}{2} \right)^2 \) or \( \pi = \left( \frac{4}{81} \right) (64) = 3.16049382 \).

The general case for a circle with diameter \( d \) would be \( \left( \frac{8}{9} d \right)^2 = \pi \left( \frac{d}{2} \right)^2 \), which leads to the same value of \( \pi \). Second, Problem 48 can be represented by the following diagram:

![Diagram of a circle and a regular hexagon](image)

By claim that (area of hexagon) = (area of circle), \( 9^2 - 4 \left[ \frac{1}{2} (3)(3) \right] = \pi \left( \frac{9}{2} \right)^2 \) or

\[ \pi = \left( \frac{4}{81} \right) (81 - 18) = 3.1111 \ldots \]

• The claim is that circumference of 30 = \( \pi \text{(diameter of 10)} \) or \( \pi = 3 \). (For more on this value, see the suggested writing topics below.)

• By Proposition 2, \( \frac{\pi^2}{(2r)^2} = \frac{11}{14} \) or \( \pi = \left( \frac{11}{14} \right) (4) = 3.14285714 \ldots = \frac{22}{7} \), which is the upper bound of Archimedes other estimate of \( 3 \frac{1}{7} \) for the circumscribed 96-gon.

• By this claim, \( \pi^2 = \left( \frac{2\pi}{2} \right) \left( \frac{2r}{2} \right) = \pi^2 \). Thus, though the correct value of \( \pi \) is involved, we do not obtain a numerical approximation of that value. But, it does provide a clever way to “calculate” the area of a circle using its corresponding measurements of circumference and diameter.
Extension 2: Using \( n, N = 3(2^{n-1}) \), \( p_n = 2N \sin \left( \frac{\pi}{N} \right) \), and \( P_n = 2N \tan \left( \frac{\pi}{N} \right) \). We have:

<table>
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<th>( n )</th>
<th>( N )</th>
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<th>( \frac{P_n}{2} )</th>
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Computing the desired limits requires L’Hopital’s Rule. For example,

\[
\lim_{N \to \infty} \frac{p_n}{2} = \lim_{N \to \infty} N \sin \left( \frac{\pi}{N} \right) = \lim_{N \to \infty} \frac{\sin \left( \frac{\pi}{N} \right)}{N^{-1}} = \lim_{N \to \infty} \frac{\cos \left( \frac{\pi}{N} \right) \left( -\pi N^{-2} \right)}{-N^{-2}} = \lim_{N \to \infty} \cos \left( \frac{\pi}{N} \right) \pi = \pi.
\]

Open-Ended Exploration: The exploration opens students’ curiosity as to what remains “constant” when certain rules change. For example, in a taxicab geometry, the value of \( \pi = 4 \), which is not only rational but also a whole number.

Teacher Commentary:

When investigating Archimedes’ approximation, remind students that he had no access to the tangent or sine functions used in modern explanations of his method. Nonetheless, it provides an opportunity to speculate on whether or not Archimedes was indirectly using these functions in the form of chord ratios. Also, Archimedes did not have access to decimal notation, with the fraction calculations becoming quite tedious, exemplified by Archimedes often replacement a fraction by another more amenable fraction. Extension Problem #2 explores Archimedes’s method, but using the power of trigonometry and decimals.

Archimedes’ justification as to why \( \frac{\sqrt{3}}{1} > \frac{265}{153} \) is unknown, though Heath (1912, pp. lxxx-lxxxiv) speculates on how Archimedes arrived at these values. Heath shows how the result can be obtained by repeated approximations using both the formula
\[ \sqrt{a^2 + b} = a + \frac{b}{2a} \] proposed by Heron and the formula \[ \sqrt{a^2 + b} = a + \frac{b}{2a + 1} \] proposed by the Arabian Alkarkhī (a 11th century mathematician who extended Greek sources).

This exploration provides the details of only the circumscribed case, while providing some insight as to Archimedes’s reasoning for the inscribed case. For the interested student, Heath (1912, 1956) provides a thorough presentation of the inscribed case.

Extension Problem #1 mentions numerous “ancient” approximations for \( \pi \), including Archimedes value as found in Proposition 2 of his *Measurement of a Circle*. This can be troubling, since this Proposition immediately precedes Proposition 3, which is the focus of the main problem’s exploration of circumscribed and inscribed triangles. To resolve this dilemma, Heath (1912, p. 93) claims “the text of this proposition is not satisfactory, and Archimedes could not have placed it before Proposition 3, as the approximation depends upon the result of that proposition.” Later, referring to our dependence on multiple translations and transcriptions of Archimedes’ work, Heath (1956, p. 305) suggest that “Proposition 2 is at all events not in its proper place. Perhaps we have only a fragment of a longer treatise.”

Extension Problems #2 re-interprets Archimedes’ method, viewable dynamically at the web site [http://demonstrations.wolfram.com/ArchimedesApproximationOfPi/](http://demonstrations.wolfram.com/ArchimedesApproximationOfPi/)

Also, resources such as Rosenberg (2003), Chabert (1999), Miel (1983), and [http://mathdl.maa.org/jsp/search/searchResults.jsp?url=http://mathworld.wolfram.com/ArchimedesRecurrenceFormula.html](http://mathdl.maa.org/jsp/search/searchResults.jsp?url=http://mathworld.wolfram.com/ArchimedesRecurrenceFormula.html) will guide students interested in exploring further Archimedes’ recurrence formula and the associated error.

As a good source of writing projects, students can explore any of the following ideas relative to Archimedes and his approximation of \( \pi \):

- Johann Pfaff, Gauss’ research adviser, generalized Archimedes’ approximation process into the form of a recursive algorithm. Published by Pfaff in 1800, the algorithm was formalized by Carl Borchardt further in the mid-1800’s by with his work with algebraic and geometric means. Investigate the mathematics underlying what is now known as the Borchardt-Pfaff algorithm. Miel (1983) is a starting resource.

- Borwein (2006, p. 3) suggests a pictorial proof of Archimedes’ approximation of \( \pi \), where the obtained digits are shaded modulo ten. Investigate the use of this interesting technique in other approximation situations.

- Extension Problem #1 reveals a Biblical value of \( \pi = 3 \). Investigate the religious impact of this “incorrect” value. For example, Belaga (2000) discusses a “Rabbinical exegesis” that re-interprets this same verse to obtain a Hebrew-value of \( \pi = 3.141509 \ldots \). And, for other interesting interpretations of the incorrectness of \( \pi = 3 \) that “gives Bible detractors such glee,” see O'Shaughnessy (1983), [http://www-groups.dcs.st-and.ac.uk/~history/Extras/Graf_theory.html](http://www-groups.dcs.st-and.ac.uk/~history/Extras/Graf_theory.html), [http://www.icr.org/article/524/](http://www.icr.org/article/524/), [http://www.purplemath.com/modules/bibleval.htm](http://www.purplemath.com/modules/bibleval.htm), or
Investigate the myriad of other geometric and analytic techniques for approximating the value of pi. Borwein (2006), Berggren et al (1997), and Beckmann (1976) are great resources in guiding this investigation.

Though Archimedes never mentions a number equivalent to the number now denoted by the symbol $\pi$, though he basically found its approximate value. Investigate the history underlying the appearance and acceptance of the value $\pi$, plus the eventual development of our formula for area of circle. Two resources are Beckmann (1976) and http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Pi_through_the_ages.html.

**Additional References:**


