Archimedes and His Quadrature of a Parabola

Solution Commentary:

Solution of Main Problems:

1. As constructed, parallelogram RMVW has RW = MV = \frac{1}{2}QV. By the Theorem and substitution, \(PV:PW = QV^2:RW^2 = (2RW)^2:RW^2 = 4RW^2:RW^2 = 4\) or \(PV = 4PW\). But, \(RM = MV = \frac{3}{4}PV\) implies \(PV = \frac{4}{3}RM\).

2. The justifications for each step in Archimedes’s proof for Proposition B are:
   - Point Y is the midpoint of PQ because RY is the diameter for chord PQ.
   - Point M is the midpoint of QV because RM is parallel to PV, which means \(\triangle QYM \sim \triangle QPV\) and corresponding sides are proportional.
   - \(PV = \frac{4}{3}RM\) because of Proposition A.
   - Also \(PV = 2YM\) because \(\triangle QYM \sim \triangle QPV\) and corresponding sides are proportional, where \(QP = 2QY\).
   - Therefore \(YM = 2RY\) because \(PV = \frac{4}{3}RM = 2YM\) implies \(\frac{2}{3}RM = YM\) or \(YM = 2RY\).
   - And \(\text{area}(\triangle PQM) = 2 \text{area}(\triangle PRQ)\) because of some proportional relationships not mentioned by Archimedes. Construct altitudes from R and M to PQ at points X and Z respectively, forming similar triangles \(\triangle YRX\) and \(\triangle YMZ\) by AAA. Then, because corresponding sides are proportional, \(YM = 2RY\) implies \(MZ = 2RX\) and \(\text{area}(\triangle PQM) = 2 \text{area}(\triangle PRQ)\) since both triangles have the same base PQ.
   - Hence \(\text{area}(\triangle PQV) = 4 \text{area}(\triangle PRQ)\) because \(MV = MQ\) implies \(\text{area}(\triangle PQV) = 2 \text{area}(\triangle PQM)\).
   - And \(\text{area}(\triangle PQq) = 8 \text{area}(\triangle PRQ)\) because \(qV = VQ\) implies \(\text{area}(\triangle PQq) = 2 \text{area}(\triangle PQV)\).

3. Using the hint, \(b=\frac{1}{3}B, c=\frac{1}{3}C, d=\frac{1}{3}D, \text{etc.},\) which implies that \(B+b = \frac{4}{3}B = \frac{4}{3}\left[\frac{1}{4}A\right] = \frac{1}{3}A\). Similarly, \(C+c = \frac{1}{3}B\), etc. Thus, the sum

\[
B+C+D+\ldots+Z+b+c+d+\ldots+z = \frac{1}{3}(A+B+C+\ldots+Y).
\]

But, subtracting \(b+c+d+\ldots+y = \frac{1}{3}(B+C+D+\ldots+Y)\) from the respective sides of the sum, \(B+C+D+\ldots+Z+z = \frac{1}{3}A\). Finally, by adding \(A\) to both sides and substituting \(z=\frac{1}{3}Z\), we obtain \(A+B+C+\ldots+Z+\frac{1}{3}Z = \frac{4}{3}A\).
Extension 1: \[ \Delta PQq + \Delta PQq + \Delta PQq + \Delta PQq + \ldots = \Delta PQq \left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots\right) \]

But, the infinite geometric series \( \left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots\right) \) with constant ratio \( \frac{1}{4} \) has sum \( \frac{1}{1 - \frac{1}{4}} = \frac{4}{3} \).

But, the sum \( \Delta PQq + \frac{\Delta PQq}{4} + \frac{\Delta PQq}{4^2} + \frac{\Delta PQq}{4^3} + \ldots \) also approximates the area of the segment PQq geometrically (i.e. the \( \frac{1}{2}Z \) term goes to zero as the “new” triangles are added an infinite number of times). Thus, \( \Delta PQq + \frac{\Delta PQq}{4} + \frac{\Delta PQq}{4^2} + \frac{\Delta PQq}{4^3} + \ldots = \frac{4}{3} \Delta PQq \)

Extension 2: The situation can be represented by this graph:

![Graph of a parabola and a triangle](attachment:graph.png)

Then, area segment = \( \int_{-2}^{2} (4 - x^2) dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \left[8 - \frac{8}{3}\right] - \left[-8 + \frac{8}{3}\right] = \frac{32}{3} \). Also, area triangle = \( \frac{1}{2} (4)(4) = 8 \). And, finally, it is true that \( \frac{32}{3} = \frac{4}{3} (8) \).

Open-Ended Exploration: This task is not trivial and will require considerable ingenuity on the part of the student. A compromise is to ask students to try to reconstruct and explain Archimedes’ solution using either Heath (1953, pp. 233-246) or Dijksterhuis (1987, pp. 336-345).

Teacher Commentary:

At the beginning, it may be necessary to point out that Archimedes views a parabola as the section of a right-angled cone. Also, he allows a parabola to have multiple vertexes and diameters, in contrast to modern restrictions to one vertex and one diameter. The difficulty lies more in the semantics than the operational aspects.

Though the mathematical calculations can get quite tedious in the exploration of these problems, it is important to place the emphasis on Archimedes’ overall approach. The key idea is his use of methods of exhaustion to create an infinite sequence of triangles whose combined areas equal the segment of a parabola.
Extension Problems #1 and #2 then revisit the problem through the perspective of limits and integration.

Archimedes approach to the problem of quadrature is important as it leads into Fermat's quadrature of the parabola, complemented by the efforts of Gregory of St. Vincent. The ultimate effect is that these efforts help build an important connection between logarithmic and exponential functions.

Observant students will see the "method of exhaustion" being used to mimic our modern use of limits. Some may conclude that Archimedes' "fudging" is unwarranted, especially in his leap from Proposition C to Proposition D. Nonetheless, such "leaps" provide great opportunities to discuss the relative roles and powers of limits. For example, why does the notion \( \frac{1}{3} Z \to 0 \) imply that eventually the area of the inscribed triangle-based polygon equals the area of the segment.

As a good source of writing projects, students can explore any of the following ideas relative to Archimedes and his quadrature of a parabola:

- A part of his argument, Archimedes used double *reductio ad absurdum*. Investigate and document historical uses of this proof technique, such as by Aristotle and Euclid. Two resources are [http://www.iep.utm.edu/r/reductio.htm](http://www.iep.utm.edu/r/reductio.htm) and [http://www.math.wustl.edu/~sk/eolss.pdf](http://www.math.wustl.edu/~sk/eolss.pdf).

- Archimedes’ approach illustrates an early use of proof by exhaustion. Investigate and document the history of this proof technique, connecting it to the co-development of limit concepts. Some useful resources are Allman (1889), Gow (1968), Heath (1963), and Van der Waerden, B. (1988).

- In Archimedes’ construction, one can draw the tangents to the parabola at points Q and q, which will intersect at some point O. Investigate Archimedes study of the special triangle OQq and its relationship to the segment of the parabola. Some useful resources are Dörrie (1965) and [http://www.cut-the-knot.org/Curriculum/Geometry/ArchimedesTriangle.shtml](http://www.cut-the-knot.org/Curriculum/Geometry/ArchimedesTriangle.shtml).

- Extension Problem #1 involved showing that \( \frac{4}{3} \) is the sum of the infinite geometric series \( 1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots \). Investigate visual proofs of this sum, using Nelson (1993, 2000).

**Additional References:**


