

# Babylonian Solutions of the Quadratic

## Historical Context:

- When: ca. 4000 B.C. – 300 B.C.
- Where: Babylon
- Who: Babylonian mathematicians
- Mathematics focus: Demonstrate use of rhetorical algebra to determine the root(s) of quadratic equations.

## Suggested Readings:

- Babylonian contributions to mathematics:  
[http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Babylonian\\_mathematics.html](http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Babylonian_mathematics.html)



- NCTM's *Historical Topics for the Mathematics Classroom* (1969): “Algebraic equations and notation” (p. 234), “Babylonian algebra—Rhetorical style” (pp. 235-236), and “Babylonian numeration system” (pp. 36-38)
- Of the half-million Babylonian clay tablets that have been found, more than a thousand involve mathematics such as include fractions, algebra, geometry, and Pythagorean triples. A common example is a tablet called Plimpton 322:  
<http://www.math.ubc.ca/~cass/courses/m446-03/pl322/pl322.html>



- Key search words/phrases: Babylon mathematics, Babylon numeration, cuneiform, solving quadratic equations, rhetorical algebra

### Problem to Explore:

Find the geometric solution(s) to the quadratic equation  $x^2 + nx = m$  for positive values  $n$  and  $m$ .

### Why This Problem is Important:

- The Babylonian's approach to finding roots of quadratic equations, a common example of rhetorical algebra, revealed strong connections to an underlying geometry similar to that of al-Khwarizimi and Euclid.
- The problem illustrates how our modern use of the quadratic formula could be viewed as rhetorical algebra as well.

### Problem Solving Experiences:

It is useful to look first at the Babylonian solution of the specific case of  $n = 2/3$  or  $0;40$  and  $m = 0;35$  [i.e.  $x^2 + (0;40)x = 0;35$ ]. The quadratic was generated by this verbal problem: "The area plus  $2/3$  a side of a square is  $0;35$ . Find the side of the square." The problem clearly suggests an underlying geometry via the [area of a square with unknown side length  $x$ ] + [area of a rectangle with side lengths  $x$  and  $0;40$ ] = [area of some shape equal to  $0;35$ ].

Note: To avoid ambiguity in place values in this discussion, the semi-colon (;) is used to represent the shift from whole to fractional place values, while the comma (,) represents a single change in place value. For example, the number  $3,2;4,6$  is equivalent to the base-60 number in expanded notation of  $(3 \times 60^3) + (0 \times 60^2) + (0 \times 60^1) + (2 \times 60^0) + (4 \times 60^{-1}) + (6 \times 60^{-2})$ , or 648002.0683 in base ten.

Using a rhetorical style of algebra, the Babylonians suggested verbal steps were:

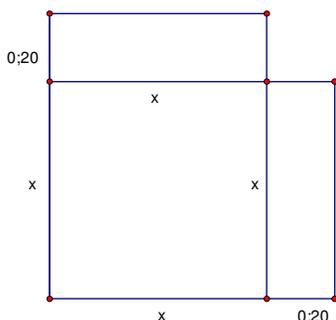
- Half of  $0;40$  is  $0;20$
- Multiply  $0;20$  by itself to get  $0;6,40$
- Add product to  $0;35$  and result is  $0;41,40$
- Take square root of  $0;41,40$  to get  $0;50$
- Subtract  $0;20$  getting result of  $0;30$

That is, the Babylonian approach is claiming that not only does  $(0;30)^2 + (0;40)(0;30) = 0;35$ , but also that this verbal recipe could serve as the model for solving any quadratic equation of the form  $x^2 + nx = m$ .

1. Does  $(0;30)^2 + (0;40)(0;30) = 0;35$ ? Try to do the computations first in base 60 and then in base ten.
2. Perform the same sequence of verbal steps in trying to "solve" the general case of  $x^2 + nx = m$ . Explain how this is related to the quadratic equation used today.

3. To show the geometry involved (something the Babylonians did not do), interpret the previous verbal steps in terms of this sequence of geometrical actions:

- Draw a square of area  $x^2$  with side lengths  $x$
- Divide the rectangle of side lengths  $x$  and  $(0;40)$  into two congruent rectangles of side lengths  $x$  and  $(0;20)$
- Attach these two new rectangles to the sides of the square, as shown:



- To “complete the square,” add a small square of side length  $(0;20)$  to produce an [square with an area of  $x^2 + (0;40)x + (0;20)^2$ ], which must be equivalent to a [square area of  $[0;35 + (0;20)^2 = 0;41,40]$ .
- But the large square has side length  $(x + 0;20)$  by construction, which must equal  $0;50$  (or  $\sqrt{0;41,40}$ ), Thus,  $x = 0;30$ .

Given this geometrical interpretation, will the Babylonian’s approach always produce a positive root for the general case of the quadratic equation  $x^2 + nx = m$  for positive values  $n$  and  $m$ ? Explain.

4. Perform this same construction of “completing the square” on the general quadratic equation  $ax^2 + bx + c = 0$ , to produce the standard quadratic formula.

### Extension and Reflection Questions:

Extension 1: The Babylonians also solved the verbal problem: “The area of a square, less the side, is 14,30. Find side.” That is, they were trying to solve the quadratic equation  $x^2 - x = 14,30$ . Their rhetorical approach was:

- Take half of 1, which is  $0;30$ , and square it, giving  $0;15$
- Add this to  $14,30$  to get  $14,30;15$
- Square root of  $14,30;15$  is  $29;30$
- Add  $0;30$  to  $29;30$  to get  $30$ , length of side

First, check that their answer is correct. Second, modify the process shown in problem #3 to create the geometrical actions that “justify” their recipe. And third, will the general quadratic equation of  $x^2 - nx = m$  always have a positive root? Explain.

Extension 2: As a slight variation, the Babylonians also used rhetorical algebra to solve this problem (stated in base 10): “Find two numbers whose product is 96 and sum is 20.” This problem creates a quadratic situation since  $xy = 96$  and  $x + y = 20$  imply by substitution that  $x(20-x) = 96$  or  $x^2 - 20x + 96 = 0$ . Their verbal recipe was as follows:

- Divide by two the sum of the numbers to get 10
- Square this result  $10^2 = 100$
- Subtract given area from this, or  $100 - 96 = 4$
- Take the square root, 2
- The base is  $10 + 2 = 12$
- The height is  $10 - 2 = 8$

Based on the wording of the solution process, how might the Babylonians connect this problem statement to land measurement? Explain. Also, perform the same sequence of verbal steps in trying to “solve” the general case of  $x^2 - nx + m = 0$  and is there always a positive root for this situation, given  $n$  and  $m$  positive values? Explain. Finally, try to create the geometrical actions that “justify” their recipe, which is closely tied to the standard algebraic identity  $a^2 - b^2 = (a-b)(a+b)$ .

Extension 3: Relooking at the previous extension problem, the Babylonians had another approach to solving  $xy = m$  and  $x + y = n$  (or  $x^2 - nx + m = 0$ ). Though quite similar to the verbal recipe already given, it has a slight twist:

- Divide  $n$  by two to get  $(n/2)$
- Put  $x = (n/2) + t$  and  $y = (n/2) - t$ , where  $t$  basically is an “error” term
- Then  $xy = m$  becomes  $(n/2)^2 - t^2 = m$

- And  $t = \sqrt{\left(\frac{n}{2}\right)^2 - m}$

- Then  $x = (n/2) + \sqrt{\left(\frac{n}{2}\right)^2 - m}$  and  $y = (n/2) - \sqrt{\left(\frac{n}{2}\right)^2 - m}$

In addition to using a clever substitution, the Babylonians also showed knowledge of an interesting relationship. Rewriting the last step by substituting for  $n$  and  $m$ , we get

that  $x = \left(\frac{x+y}{2}\right) + \sqrt{\left(\frac{x+y}{2}\right)^2 - xy}$  and  $y = \left(\frac{x+y}{2}\right) - \sqrt{\left(\frac{x+y}{2}\right)^2 - xy}$ . This looks

complicated, but the Babylonians knew that  $\sqrt{\left(\frac{x+y}{2}\right)^2 - xy} = \left(\frac{x-y}{2}\right)$ . Use algebra to

show that this relationship is true, then show that  $x = \left(\frac{x+y}{2}\right) + \left(\frac{x-y}{2}\right)$  and

$$y = \left(\frac{x+y}{2}\right) - \left(\frac{x-y}{2}\right).$$

Open-ended Exploration: The Babylonian’s approach only produced at most one positive root for a given quadratic equation. Can you make adjustments that will also produce the negative roots? The complex roots?