Fight Over Solving the Cubic

Historical Context:
- When: 1535
- Where: Republic of Venice (now Italy)
- Who: Primarily Niccolo Fontana aka Tartaglia and Girolamo Cardano
- Mathematics focus: Demonstrate development of a formula for solving the cubic equation.

Suggested Readings:
- Tartaglia’s life and contributions to mathematics: http://www-history.mcs.st-and.ac.uk/Biographies/Tartaglia.html
- Cardano’s life and contributions to mathematics: http://www-history.mcs.st-andrews.ac.uk/Biographies/Cardan.html
- Quadratic, Cubic, and Quartic Equations http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Quadratic_etc_equations.html
- The cubic dispute between Tartaglia and Cardano http://www-history.mcs.st-andrews.ac.uk/HistTopics/Tartaglia_v_Cardan.html
- NCTM's Historical Topics for the Mathematics Classroom (1969): “Solution of polynomial equations of third and higher degrees” (pp. 276-279) and “Algebra in Europe, 1200-1850” (pp. 309-311)
- Key search words/phrases: Tartaglia, Cardano, Fior, del Ferro, cubic, formula, equation fight, imaginary numbers, Galois, Abel

Problem to Explore:
Solve the cubic equation \( x^3 + 6x = 20 \) using an unusual substitution method.
Why This Problem is Important:

- Important step in generation of a general formula for solving the cubic equation.
- Forced mathematicians to confront the possibility of imaginary numbers being solutions to polynomial equations, though situation was not resolved.

Problem Solving Experiences:

1. Consider the cubic equation $x^3 + 6x = 20$. How many real roots could the equation have? Using a graphing calculator, graph the function $f(x) = x^3 + 6x - 20$ using the window $-5 \leq x \leq 5$ and $-5 \leq y \leq 5$. What root(s) are found by inspection?

In the 1535 contest with Fior, Tartaglia had to solve the equation *1.cubo piu .6.cose, equal à .20.*, which in modern notation is the same as our $x^3 + 6x = 20$. His approach is as follows:

- Let $x = u - v$, which by substitution implies that $(u-v)^3 + 6(u-v) = 20$ or $u^3 - 3u^2v + 3uv^2 - v^3 + 6u - 6v = 20$
- Choose product $uv = 1/3$ coefficient of x-term, or $uv = (1/3)(6) = 2$
- Substitute $uv = 2$ into the cubic expression involving $u$ and $v$ to get $u^3 -3(2)u +3(2)v - v^3 + 6u -6v = 20$, which reduces to $u^3 - v^3 = 20$
- But, $v = 2/u$ implies that $u^3 - 8/u^3 = 20$ or $(u^3)^2 - 20(u^3) - 8 = 0$
- Since this can be solved as a quadratic, we have $\frac{u^3 = 20 \pm \sqrt{400 + 32}}{2} = \frac{20 \pm \sqrt{108}}{2} = 10 \pm \sqrt{108}$ or $u = \sqrt[3]{10} \pm \sqrt[3]{108}$

2. Given the sign options, four different possibilities need to be considered to find a numerical value for the root $x$. Calculate all four “roots.” Do any of them agree with your answer in Problem #1? Are all of them solution roots for the equation $x^3 + 6x = 20$? Explain your reasoning.

3. Use Tartaglia’s technique to solve the equation *1.cubo piu .3.cose, equal à .10*. Which of the four possibilities are the “real” root(s)? What does the graph of the associated function reveal?

Extension and Reflection Questions:

**Extension 1:** Occasionally surprises would occur using this substitution technique. For example, use it to solve $x^3 = 15x + 4$. What is the surprise? What does the graph of the associated function show as to the number of real root(s) and their value(s)?
Extension 2: In the previous problem, we can determine by inspection that $x = 4$ is a solution of $x^3 = 15x + 4$. But, how does this information fit with your derived root(s)? Hint: Assume the two cube root expressions are both complex (i.e. let them equal $m+ni$ and $p+qi$ respectively) and try to find the respective values of $m$, $n$, $p$, and $q$.

Extension 3: To establish a closed formula for solving the general cubic equation, several steps are needed:

- Justify the algebraic identity $(u-v)^3 + 3uv(u-v) = u^3 - v^3$.
- Show how this identity is the key to Tartaglia’s method for solving the generalized equation $x^3 + mx = n$.
- Show how this now leads not only to the two equations $u = \sqrt[3]{\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}$ and $v = \sqrt[3]{\frac{n}{2} - \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}$, but also to the classical Cardano-Tartaglia formula of $x = \sqrt[3]{\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} - \sqrt[3]{\frac{n}{2} - \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}$.
- Finally, show that the transformation $y = x - b/(3a)$ converts the general cubic equation $ay^3 + by^2 + cy + d = 0$ into the desired form $x^3 + mx = n$.

Open-ended Exploration: In the quadratic formula for solving the equation $x^2 + bx + c = 0$, the expression $b^2 - 4c$ is called the discriminant and gives information on the existence of 2 real roots ($b^2 - 4c > 0$), 1 double root ($b^2 - 4c = 0$), and 2 complex roots ($b^2 - 4c < 0$). Investigate the classical Cardano-Tartaglia formula to determine which expression plays the role of a discriminant and its information relative to the root possibilities.