Descartes’ Geometrical Solutions of the Quadratic

Solution Commentary:
Elaboration on Hint: The useful geometry theorem is:

If a secant segment AD and tangent segment AB are drawn to a circle from the same external point A, the product of the length of the secant segment AD and its external part (AC) equals the square of the length of the tangent segment AB. That is, \((AD)(AC) = (AB)^2\).

Solution of Main Problem: By the geometrical theorem involving tangents and secants to a circle from a point, \((BE)(BF) = (AB)^2\). Now \(BF = BE + n\) because \(EF = AC = n\). Thus, by substitution, we get \((BE)(BE + n) = (m)^2\) or \((BE)^2 + n(BE) = (m)^2\).

Alternate Solution of Main Problem: Since triangle ABD is a right triangle by construction, \((DA)^2 + (AB)^2 = (BD)^2\). Then, by substitution, \((\frac{n}{2})^2 + m^2 = \left(BE + \frac{n}{2}\right)^2\) which can be manipulated algebraically into the desired relationship \((BE)^2 + n(BE) = (m)^2\).

Extension 1: Since \(m > 0\), the two relationships \(x=-\frac{n \pm \sqrt{n^2+4m^2}}{2}\) and \(\sqrt{n^2+4m^2} > \sqrt{n^2} = n\) imply that \(x=-\frac{n + \sqrt{n^2+4m^2}}{2} > 0\) and \(x=-\frac{n - \sqrt{n^2+4m^2}}{2} < 0\).

Extension 2: By the same geometry theorem, \((BE)(BF) = (AB)^2\), which implies by substitution that \((BF - n)(BF) = (AB)^2\) or \((BF)^2 = n(BF) + (AB)^2\). Again, there is always one positive root and one negative root.
Extension 3: The same geometry theorem can still be used to obtain the same relationship of $(BE)(BF) = (AB)^2$. But, to simplify things before proceeding, draw a line through point C parallel to segment AB, intersecting line BE at point G:

Then, by symmetry relationships, $GF = BE$ and $BG = AC = n$. If $BF$ was a root, then substitution in $(BE)(BF) = (AB)^2$ becomes $(n - BF)(BF) = (AB)^2$ or $(BF)^2 + (AB)^2 = n(BF)$. Similarly, if $BE$ was a root, then substitution in $(BE)(BF) = (AB)^2$ becomes $(BE)(n - BE) = (AB)^2$ or $(BE)^2 + (AB)^2 = n(BE)$. If $AB = AD$ (or $m = n/2$), then points E and F are the same point as the line BE is now tangent to circle W, which means BE is a double root. Finally, if $AB > AD$ (or $m > n/2$), then points E and F do not exist as the perpendicular line does not intersect circle W, which means no real roots exist. Note how much powerful the geometry view becomes, as the similar argument for root possibilities becomes quite convoluted using algebraic manipulations of the quadratic formula.

Extension 4: A footnote in the Smith & Latham translation of Descartes The Geometry (p. 17) argues that Descartes fell short of his own goal in that he was unable to implement his method using generalized parameters (i.e. cases were built on his need to work with positive parameters n and m). Also, Descartes makes no reference to the case of $x^2 + nx + m^2 = 0$ for positive n and m values. Perhaps it was because Descartes knew this equation had no positive roots from the outset…or perhaps it was because his technique could not be applied in this case.

Open-Ended Exploration: Multiple adjustments are possible. For example, Nannini (1966) provides a great discussion as to how all of the roots (positive, negative, or complex) can be obtained by one single method abstracted from Descartes approach.

Teacher Commentary:
Ask students to implement Descartes’ process using both a straight-edge/compass and GSP. The first option of a straight-edge/compass adds a sense of reality to the idea that even though they might start with a simple values for n and m, they now face the new problem of trying to determine the numerical value of the obtained root(s). That is, the process is theoretically sound and does produce a length as a root, but algebra is needed to produce the numerical value. In contrast, the GSP option encourages students to explore how changes in the starting values of n and m affect the solution root(s), especially in the third case of \(x^2 + m^2 = nx\).

As a good source of writing projects, students can explore how other authors have taken different approaches to understanding and using Descartes’ process:

- Torres-Hernández & Sosa-Garza identifies the mathematics underlying “how his method was conceived” and thereby covers some gaps.
- Patterson & Lubecke (1991) uses relationships between parabolas and circles to construct a single circle that produces the real roots of any quadratic.
- Meconi (1972) creates “Newton p-polygons” which are used to basically extend Descartes’ process of finding roots for quadratics to factoring polynomials of any degree n.

Additional References:


