Descartes’ Transform-Solve-Invert Method

Solution Commentary:

Solution of Main Problems:

1. Rewriting the problem in base ten notation, we have LXIII x XXIV = 63 x 24. The product is 1512, which is MDXII in Roman Numerals.

2. If BN is of length 0 (or B = N), then E and C are the same point. If BN > length 0, then point F is a point on segment CD, moving closer to point D as BN gets larger. Finally, be prepared to handle the following interpretation as well:

Descartes only discussed the solution to the case of length BN > 0, or where point F is between points D and C.

3. Let CD = a and DF = x, which implies that CF = a-x. Triangles CEF and DBF are similar by A.A. criterion, which implies by proportionality of corresponding sides that CF/FE = FD/BF or (a-x)/c = x/BF or BF = cx/(a-x). Then, applying the Pythagorean Theorem to right triangle BDF, we have BF^2 = BD^2 + FD^2 or (cx/(a-x))^2 = a^2 + x^2. Clearing the denominator and simplifying terms, we end up with the fourth-degree equation x^4 - 2ax^3 + 2a^2x^2 - 2a^3x + a^4 = c^2x^2.

Descartes then expressed the solution for a as follows:

\[ x = \frac{1}{2}a + \sqrt{\frac{1}{4}a^2 + \frac{1}{4}c^2 - \frac{1}{4}c^2 - \frac{1}{2}a^2 + \frac{1}{2}a\sqrt{a^2 + c^2}}. \]

Geometrically, this also was a “transformed” solution, in that the length x could “theoretically” be constructed using a straight-edge and compass because at most square roots are involved. (See commentary note below.)

Extension 1: Segment BD parallel to segment CE implies that corresponding angles are congruent, making triangles ABD and ACE similar by AAA. Then, due to proportionality of sides of similar triangles, we have AB:AC = AD:AE. Finally, AB:AC = AB:(AB+BC) = AD:AE = AD:(AD+DE). Inversion of the proportions implies that (AB+BC):AB = (AD+DE):AD, or that 1 + BC:AB = 1 + DE:AD. Cancelling the common unit, we have BC:AB = DE:AD, which can be inverted to be AB:BC = AD:DE.
Extension 2: By Extension Problem #1, we have $1:m = n:DE$, which implies that segment DE has length $mn$. The associated proportions for $m/n$ and $\sqrt{m}$ are $m:n = 1:[a/b]$ and $1:\sqrt{m} = \sqrt{m}:m$. The corresponding diagrams used by Descartes are:

In the diagram for the $\sqrt{m}$ case, a semicircle is constructed on diameter of length $1+m$. The perpendicular at point A makes triangles ABD and ADC similar, and thus $AB:AD = AD:AC$.

Extension 3: This problem is taken from Pappas’ Collection (Prop. 72, Vol. II, p. 783). The proof becomes short once Pappus established a Lemma: Given a square $ABDC$, and $E$ a point in $AC$ produced, $EG$ perpendicular to $BE$ at $E$, meeting $BD$ produced in $G$, and $F$ the point of intersection of $BE$ and $CD$. Then $CD^2 + FE^2 = DG^2$.

First, $EG$ perpendicular to $BD$ at point $E$ because it is part of an angle inscribed in a semicircle. And, triangle NBD a right triangle implies that $DN^2 = BD^2 + BN^2$ as claimed. But why should $CD^2 + FE^2 = DG^2$, since the three lengths are not the sides of a common right triangle….that is the substance of the Lemma. However, since the Lemma is provable, substitute $DG = DN$, $CD = BD$, and $FE = BN$ to get the final conclusion that $FE = BN$.

NOTE: The answer to the cryptic claim or Lemma is not being provided, thereby allowing your own students to perhaps find their own “clever” approach. Unless some known geometric fact is being ignored, the situation is not as “obvious” as Pappus made it appear (but then I have not seen his proof of his Lemma). Given the challenge, I have had several students produce valid proofs of the Lemma, but they certainly are not trivial.

Open-Ended Exploration: This is a new problem to explore that neither Descartes nor Pappus apparently considered. But, their techniques perhaps can be adapted. The explorations become more interesting and fruitful if students examine models using dynamic software such as GSP.

Teacher Commentary:

When investigating these problems, students need to see the strong connections being made between algebra and geometry:

- Ordered pairs $(m,n) \leftrightarrow$ Points in $x$-$y$ coordinate plane
- Equation involving two variables $x$ and $y \leftrightarrow$ Curves in a plane
- Algebraic properties of equations $\leftrightarrow$ Geometric properties of curves
  (e.g. common solution to system $\leftrightarrow$ (e.g. Intersection point of...
In Problem #3, students should end up in a quandary, having a fourth-degree polynomial that they do not know how to solve. But, Descartes had derived the proper formula for solving such an equation (see Descartes’ The Geometry, pp. 168-188); you should share the final expression for x with students, as they will not know this formula nor his techniques. Also, stress that Descartes has a theoretical-but-exact solution, yet it would be quite difficult and tedious to actually construct.

As to Extension Problem #3, if your students produce a rather straight-forward argument for the cryptic step, please share it with me. Based on the arguments constructed by my students, I am convinced that there has to be a simpler argument available to Pappus at that time (especially since the step was written as being obvious to him).

As a good source of writing projects, students can explore how other authors have taken different approaches to understanding and using Descartes’ Transform-Solve-Invert method:

• In his Analytical Geometry (1843), Auguste Comte suggests:
  *The spirit of the geometry of antiquity was essentially synthetic: that is to say, the various conditions of each problem were studied for the most part in their entirety. It is true that what was called geometrical analysis had been used in an accessory way; and this may be regarded as a first approach to the modern system; although the absence of algebraic conceptions, by which alone the separation of the various conditions of the problem could be fixed and pursued to its final consequences, deprived this procedure of its main value; so that by the geometers of Greece it was more preached than practiced. The spirit of modern mathematics since Descartes is to isolate the various conditions of a problem, and thus arrive at a perfectly general solution for each. It is thus, in the strictest meaning of the word, Analysis.*

  Do you agree with this claim regarding the differences between the geometry of antiquity and modern geometry since Descartes?

• Porubsky (2009) writes:
  *In a passage on Appollonius' Conics, the attempt to conceive of the product of three, four, five, six or more lines as geometrical entities, known as Pappus' Problem, Descartes devoted a major part of his own Géométrie to this, and solved it by the use of algebraic notation. Thus Descartes demonstrated that the difficulties which Pappus was unable to overcome could be got round by the use of his new algebraic method. Pappus thus came to play a catalytic, if minor, role in the founding of Cartesian analytic geometry.*

  Investigate this last claim regarding Pappus’ contributions.

• Domski (2009) claims there is a strong relationship between Descartes’ Geometry and his first metaphysics of nature as presented in Le Monde (1632). In particular, she argues that “the same standard of clear and
distinct motions for construction which allows Descartes to distinguish ‘geometric’ from ‘imaginary’ curves in the domain of mathematics is adopted in Le Monde as Descartes details God’s construction of nature.” Investigate the merits of this claim.


**Additional References:**


