Egyptians’ Solution of Algebraic Equations

Solution Commentary:

Solution of Main Problems:

1. Looking at the problem algebraically, we have \( x + \frac{1}{7}x = 19 = x\left(1 + \frac{1}{7}\right) \). Then, by the “false” guess of \( x = 7 \), \( 7 + \frac{1}{7}(7) = 8 \) or \( 7\left(1 + \frac{1}{7}\right) = 8 \). But by substitution, \( 8\left(2 + \frac{1}{4} + \frac{1}{8}\right) = 19 \) can be re-written as \( 7\left(1 + \frac{1}{7}\right)\left(2 + \frac{1}{4} + \frac{1}{8}\right) = 19 \), or by the a combination of the associative and commutative law, we get the final desired expression \( 7\left(2 + \frac{1}{4} + \frac{1}{8}\right)\left(1 + \frac{1}{7}\right) = 19 \), implying that \( x = 7\left(2 + \frac{1}{4} + \frac{1}{8}\right) = 16 + \frac{1}{2} + \frac{1}{8} \).

2. The steps would be
   - Assume (falsely?) that \( x = 14 \)
   - Then \( x + \frac{1}{7}x = 16 \) not 19 as required
   - But by division of 19 by 16, we know \( 16\left(1 + \frac{1}{8} + \frac{1}{16}\right) = 19 \)
     
     i.e.
     
     \[
     \begin{array}{ll}
     *1 & 16 \\
     1/2 & 8 \\
     1/4 & 4 \\
     *1/8 & 2 \\
     *1/16 & 1 \\
     \end{array}
     \]
     
   - Thus, the answer is \( x = 14\left(1 + \frac{1}{8} + \frac{1}{16}\right) = 16 + \frac{1}{2} + \frac{1}{8} \)

Though the process would still work, \( x = 10 \) is not a good guess, as it overcomplicates the first step. As to \( x = 0 \), the first step works, but one is then faced with trying to divide 19 by 0 (even impossible with Egyptian hieroglyphs). Finally, \( x = 70 \) is the best of these three “false” guesses, but it overcomplicates the situation somewhat, as now one is faced with dividing 19 by 70 (which can be done, but...).

Extension 1: Our underlying equation is \( x + \frac{1}{5}x = 21 \). The steps would be
   - Assume (falsely?) that \( x = 5 \)
   - Then \( x + \frac{1}{5}x = 6 \) not 21 as required
   - But by division of 21 by 6, we know \( 6\left(3 + \frac{1}{2}\right) = 21 \)
i.e. *1 6
*2 12
*1/2 3

Thus, the answer is \( x = \frac{5}{3} + \frac{1}{2} = \frac{17}{6} + \frac{1}{2} \).

**Extension 2:** First, simplify the equation \( \frac{x + \frac{2}{3} x}{\frac{1}{3} (x + \frac{2}{3} x)} = 10 \) by substituting

\( y = \frac{x + \frac{2}{3} x}{\frac{1}{3} (x + \frac{2}{3} x)} \) to get \( y - \frac{1}{3} y = 10 \). For the latter equation, the steps would then be

- Assume (falsely?) that \( x = 3 \)
- Then \( y - \frac{1}{3} y = 2 \) not 10 as required
- But by division of 10 by 2, we know 2(5) = 10
- Thus, the “temporary” answer is \( y = 3(5) = 15 \).
- But, as \( y = \left( x + \frac{2}{3} x \right) \), our new equation to solve is \( 15 = \left( x + \frac{2}{3} x \right) \).
- Again, assume (falsely?) that \( x = 3 \)
- Then \( \left( x + \frac{2}{3} x \right) = 5 \) not 15 as required.
- But by division of 15 by 5, we know 5(3) = 15
- Thus, \( x = (3)(3) = 9 \) is the desired answer.

**Extension 3:** Resnikoff & Wells (1984, pp. 52-54) suggest that this sequence of arithmetic calculations is occurring:

\[
10 \rightarrow 10/10 = 1 \rightarrow 10 - 1 = 9 \rightarrow (2/3)9 = 6 \rightarrow 6+9=15 \rightarrow (1/3)15=5 \rightarrow 15-5=10
\]

But, this sequence suggests this “similar” expression:

\[
\left[ \frac{2}{3} \left( 10 - \frac{10}{10} \right) + \left( 10 - \frac{10}{10} \right) \right] - \frac{1}{3} \left[ \frac{2}{3} \left( 10 - \frac{10}{10} \right) + \left( 10 - \frac{10}{10} \right) \right] = 10
\]

or \( x = 10 - \frac{10}{10} = 9 \) as the solution.

**Extension 4:** Using this same approach, we would expect to use this sequence of arithmetic calculations:

\[
20 \rightarrow 20/20 = 1 \rightarrow 20 - 1 = 19 \rightarrow (2/3)19 \ldots \text{Looks like trouble is ahead!}
\]

However, this modified sequence of arithmetic calculations:

\[
20 \rightarrow 20/10=2 \rightarrow 20-2=18 \rightarrow (2/3)18=12 \rightarrow 12+18=30 \rightarrow (1/3)30=10 \rightarrow 30-10=20
\]

which suggests this “similar” expression:

\[
\left[ \frac{2}{3} \left( 20 - \frac{20}{10} \right) + \left( 20 - \frac{20}{10} \right) \right] - \frac{1}{3} \left[ \frac{2}{3} \left( 20 - \frac{20}{10} \right) + \left( 20 - \frac{20}{10} \right) \right] = 20
\]

or \( x = 20 - \frac{20}{10} = 18 \) as the solution. Thus, to solve the general case, we perhaps would expect to this modified sequence of arithmetic calculations:
\[ \frac{n}{10} = ? \rightarrow 20 - ? = \text{ etc.} \]

which suggests this “similar” expression:

\[
\left[ \frac{2}{3} \left(n - \frac{n}{10}\right) + \left(n - \frac{n}{10}\right) \right] - \frac{1}{3} \left[ \frac{2}{3} \left(n - \frac{n}{10}\right) + \left(n - \frac{n}{10}\right) \right] = n
\]

or \[ x = n - \frac{n}{10} = \frac{9}{10} n \] as the solution, which is the same conclusion reached when modern algebra is used to solve the equation \( \left(x + \frac{2}{3} x\right) - \frac{1}{3} \left(x + \frac{2}{3} x\right) = n \) for \( x \).

**Open-Ended Exploration:** Part of the problem with the “method of false position” is that a “poor” first guess can lead to difficulties in division. Also, the Egyptian problems were quite contrived so that the first guess usually worked out nicely. Perhaps this is why the scribes’ techniques varied from problem to problem, as the solutions match the problems and vice versa. Some good resources are Resnikoff & Wells (1984), Joseph (2000), Gillings (1982), Friberg (2005), Friberg (2007), and Robins, G. and Shute (1987).

**Teacher Commentary:**

When discussing these problems, show students translated copies of the Rhind Mathematical Papyrus. Chace (1978 reprint) and Robbins & Shute (1987) perhaps are the two standard resources. Students will gain an appreciation for the translation effort when they see pictures of the actual scroll, as well as the original hieratic text, the hieroglyphic transcription, the transliteration, and the literal translation.

Stress that the *Rhind Mathematical Papyrus* is a collection of problems with their solutions. Mathematics historians tend to suggest that the Egyptian scribes were not trying to provide generalized problem-solving techniques, even though there is some rhyme-or-reason to the arrangement of the problems. At best, the conclusion is that “this method” solves “this problem.” Nonetheless, Extension Problems #2-#4 are directed at showing students just how close the Egyptians were to providing generalized problem-solving techniques.

Prior to confronting these problems, students should be comfortable with the use of unit fractions as well as multiplication and division by Egyptian methods. Most mathematics history texts include descriptions of these three ideas, or examples can be found online (e.g. [http://en.wikipedia.org/wiki/Egyptian_mathematics](http://en.wikipedia.org/wiki/Egyptian_mathematics)).

As a good source of writing projects, students can explore any of the following ideas relative to the Egyptian solution of equations:

- These problem explorations suggest that the Egyptians used the “method of false position.” Mathematics historians are divided as to the accuracy of this claim. For example, in an e-mail note sent to the Math-History-Listserve (May 22, 2006), Milo Garner writes: *As everyone that has read any serious review of Egyptian algebra, going back 80 years or more, the topic of “false position”...*
gets quickly interjected as assisting scribes to find solutions to unknowns (x). Well, it turns out that Ahmes and other scribes did not use any “false supposition,” or guesses…. Investigate the merits and arguments on both sides of the argument and present your well-documented position.

- Resnikoff & Wells (1984, pp. 54-55) claimed that the scribes’ solution to RMP Problem 28:
  
suggests that the Egyptians lacked a systematic method for solving linear equations with one unknown, which is true in the same sense that they lacked an alphabetic writing system. Just as their cumbersome system of writing included all of the ingredients for a successful method of alphabetic writing so did their collection of mathematical techniques include a general procedure for solving linear equations in one unknown…but in neither case did they recognize that their systems included special or superfluous techniques.

Do you agree with this claim and interpretation of the Egyptians’ solution of RMP Problem 28?

- The Egyptians’ solution of equations is the first documented “stop” in the long road from rhetorical algebra to syncopated algebra to symbolic algebra. Thus, the Egyptians should have provided a foundation for future algebraic techniques. Did the Babylonians, Greeks, Hindus, Muslims, and Chinese build on the Egyptian’s ideas? If not, what were the barriers in providing such a foundation? Four good references are Resnikoff & Wells (1984), Joseph (2000), Friberg (2005), and Friberg (2007).

- In his discussion of the merits of Egyptian and Babylonian mathematics, Kline (1962, p. 14) pulls no punches:
  
  Compared with the accomplishments of their immediate successors, the Greeks, the mathematics of the Egyptians and Babylonians is the scrawling of children just learning to write as opposed to great literature…Egyptian and Babylonian mathematics is best described as empirical and hardly deserves the appellation mathematics…Some flesh and bones of concrete mathematics were there, but the spirit of mathematics was lacking.

Do you agree or disagree with these claims? Elaborate.

- The “method of false position” or regula falsi method has become an important technique as part of numerical analysis. Investigate and document a history of the method, including the contributions of the Indian mathematician Vaishal (ca. 3rd century BC), the ancient Chinese text The Nine Chapters on the Mathematical Art (ca. 200 BC to AD 100), the Italian mathematician Leonardo of Pisa or Fibonacci (1202), and possibly Newton via his bisection method.

**Additional References:**


