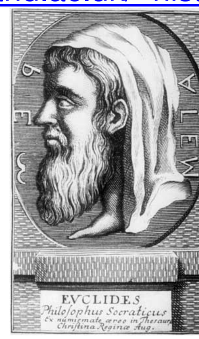
Euclid's Creation of Golden Section

Historical Context:

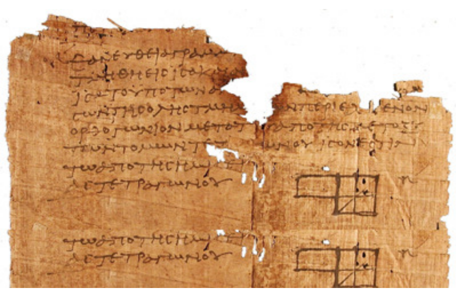
- When: 287 - 212 B.C.
- Where: Syracuse, Sicily (Greece)
- Who: Archimedes
- Mathematics focus: Investigation of the calculation of a value of pi.

Suggested Readings:

- Euclid's life and contributions to mathematics:
  http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Euclid.html



- NCTM's *Historical Topics for the Mathematics Classroom* (1969): "Later Greek geometry" (pp. 174-175) and "The Golden Section" (pp. 204-207)
- Euclid's *Elements*, being one of the oldest Greek mathematical texts still available,   established axiomatics as a standard for more than 2000 years.
  http://en.wikipedia.org/wiki/Euclid's_Elements



- Key search words/phrases: Euclid, Greek geometry, phi, golden section, golden ratio

Problem to Explore:

Investigate the use of the method of exhaustion to approximate a value of pi, sometimes known as Archimedes' constant..

Why This Problem is Important:

- Perhaps is the first definition of the Golden Ratio, though no numerical value is associated.

- Established basis for future investigations involving either the discovery of the Golden Ratio in nature or the use of the Golden Ratio in art, architecture, and design.

Problem Solving Experiences:

In Euclid's *Elements*, consider Definition 3 (Book VI): *A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less.*

Given a line segment "cut" into two parts. Then, by the definition, the line has been "cut in extreme and mean ratio" if $\frac{x}{y} = \frac{x+y}{x}$.

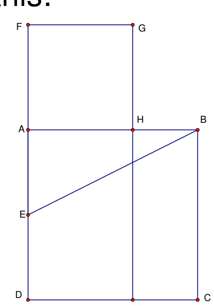
1. Assume that y = 1 and the line segment's length is x + 1. Find the numerical values (both exact and approximate) for x.

The equation $\frac{x}{y} = \frac{x+y}{x}$ can be rewritten as $x^2 = y(x+y)$. This relationship creates a visual equality between the area of a square and a rectangle, yet much more is involved as revealed in Proposition 11 (Book II): *To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.*

To explore the visual aspect of this proposition, complete these steps using either a straight-edge/compass or GSP…

- Construct a segment AB of an arbitrary length
- Construct a square ABCD on segment AB with side length AB.
- Bisect segment AD to create midpoint E
- Construct segment EB
- Locate point F on ray CA such that EF = EB
- Construct a square AFGH on segment AF with side length AF and point H on segment AB
- Let point K be the intersection of ray GH and segment DC

The diagram should now look like this:



2. Using Euclid's construction, let length AB = x. Using algebra and the implied geometry, prove that the area of square AFGH is equal the area of the rectangle HBCK.

Euclid proved the same relationship geometrically, but his proof depends on Proposition 6 (Book II), whose proof depends on three other Propositions from Book I, etc. As these Propositions are not germane to the current investigation, an algebraic proof will suffice.

Though Proposition 11 (Book II) precedes Definition 3 (Book VI) in the *Elements*, it essentially shows how to divide a line segment into a golden ratio. Given the construction of previous diagram, point H is the desired division or "cut" point, as now $(AH)^2 = (HB)(BC) = (HB)(AH+HB)$ or $\frac{AH}{HB} = \frac{AH + HB}{AH}$.

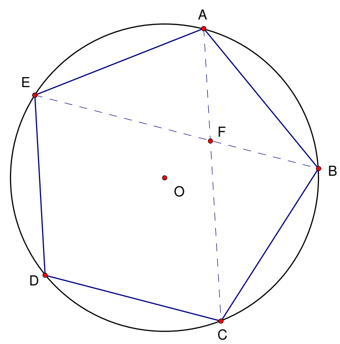
Subsequent to Definition 3 (Book VI), Euclid's offered a method for constructing the necessary "cut" in Proposition 30 (Book VI): *To cut a given finite straight line in extreme and mean ratio.* Unfortunately, the construction process is much more complicated than the process given for Proposition 6 (Book II), plus it depends on explorations of the "proportionality" content in Propositions 25 and 29 (Book VI).

Extension and Reflection Questions:

Extension 1: In the *Elements*, Propositions 1 – 18 (Book XIII) explore aspects of this "extreme and mean ratio" or Golden Section. For example, consider Proposition 5: *If a straight line be cut in extreme and mean ratio, and there be added to it a straight line equal to the greater segment, the whole straight line has been cut in extreme and mean ratio, and the original straight line is the greater segment*. If AB is the original segment with "cut" C and added segment AD =AC, the appropriate diagram is:

Given the claim in Proposition 5, prove $\frac{AB}{AD} = \frac{AB + AD}{AB}$.

Extension 2: Consider Proposition 8 (Book XIII): *If in an equilateral and equiangular pentagon straight lines subtend two angles taken in order, they cut one another in extreme and mean ratio, and their greater segments are equal to the side to the side of the pentagon.*

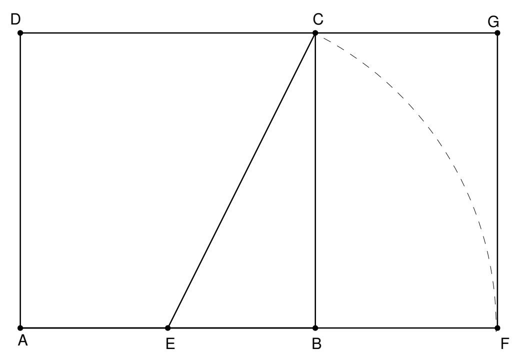


Given the Proposition's claim, prove $\frac{EF}{FB} = \frac{EF + FB}{EF}$ and EF = CF = AB. <u>Hint:</u> Show ΔEAB ~ ΔAFB.

<u>Extension 3:</u> The standard method for constructing a "golden ratio" is as follows:

- On given line segment AB, construct a square ABCD with side AB
- Bisect segment AB to get midpoint E
- Construct segment EC
- Construct a circle with radius EC and center E, intersecting ray AB at point F
- Let point G be the intersection of ray DC and the perpendicular line to segment AF at point F

The diagram, or "golden rectangle", should now look like this:



If AB = 1, prove:

- $\frac{AB}{BF} = \frac{AB + BF}{AB} = \frac{1+\sqrt{5}}{2}$, or that B is the "golden cut" for segment AF
- Given the simplicity of this construction, why doesn't it satisfy for the construction used for Proposition 11 (Book II)?

<u>Open-ended Exploration:</u> Given Proposition 8 (Book XII) in Extension Problem #2, try to create a sequence of steps for constructing a regular pentagon using only a straight edge and compass. Investigate other construction approaches as well.