Fermat’s Quadrature of Hyperbolas and Parabolas

Historical Context:
- When: 1636-1658
- Where: Toulouse, France
- Who: Pierre de Fermat
- Mathematics focus: Investigation of Fermat’s use of geometrical proportions and infinite series to determine the area between both hyperbolas and parabolas and an axis.

Suggested Readings:
- Fermat and his contributions to mathematics: [http://www.gap-system.org/~history/Biographies/Fermat.html](http://www.gap-system.org/~history/Biographies/Fermat.html)
- NCTM’s *Historical Topics for the Mathematics Classroom* (1969): “Pierre de Fermat” (pp. 410-413) and “The history of calculus” (pp. 376-391)
- Key search words/phrases: Fermat, geometrical proportion, geometric series, quadrature, hyperbola, parabola, integration, adequation

Problem to Explore:
Investigate the use of geometric proportions and infinite series to determine areas between the hyperbola $y = \frac{1}{x^2}$ and the x-axis.

Why This Problem is Important:
- Illustrates the early use of intervals, rectangles, and infinite series to determine areas.
- Is an important step in the development of finding areas as an early form of integration in calculus.

Problem Solving Experiences:
Fermat’s solution to the quadrature problem both capitalized on and extended the previous efforts of Archimedes, Cavalieri, Kepler, Oresme, and Wallis. His unique
contribution was that he was the first to perform quadrature of nonrectilineal figures over an infinite region.

Fermat began by considering the general hyperbola DSEF bounded by asymptotes AR and AC and pictured as follows:

![Hyperbola Diagram]

The hyperbolic curve is defined by the proportional relationship \( \frac{AH^n}{AG^n} = \frac{EG^m}{HI^m} \), which implies in modern notation that \( x^m y^n = k \). In the discussion that follows, the specific proportion explored is \( \frac{AH^2}{AG^2} = \frac{EG^1}{HI^1} \), representing the equation \( y = \frac{k}{x^2} \). Fermat's goal is to determine the rectilinear area of the infinite region DEGR.

After selecting points G, H, O, ... on the x-axis, Fermat constructed ordinate lines EG, IH, NO, ... parallel to asymptote AC, all guided by these two criteria:

- Segments AG, AH, AO, etc. form an infinitely increasing geometric sequence such that \( \frac{AG}{AH} = \frac{AH}{AO} = \frac{AO}{AM} = ... \).
- Segments AG, AH, AO, et.c are “close enough to each other” so that the circumscribed rectilinear parallelograms approximated the general quadrilateral (i.e. rectangle EH approximated trapezoid EGHI).

Fermat's intent was then to apply Archimedes' method of exhaustion to determine a final area. But first, he needed to establish some useful proportions.

1. Show that \( \frac{AG}{AH} = \frac{GH}{HO} = \frac{HO}{OM} = ... \).

2. Show that \( \frac{AH^2}{AG^2} = \frac{AO}{AG} \).

3. Focus on the ratio of the areas of the first two rectangles and show that \( \frac{EG \times GH}{HI \times HO} = \frac{AO}{AH} \).

By the defining proportional relationship for the segments AG, AH, and AO, \( \frac{AG}{AH} = \frac{AH}{AO} \), which implies that \( \frac{EG \times GH}{HI \times HO} = \frac{AO}{AH} = \frac{AH}{AG} \). Fermat then noted that he
similarly could prove \( \frac{HI \times HO}{NO \times MO} = \frac{AO}{AH} = \frac{AH}{AG} \), implying that “the infinitely many parallelograms \( EG \times GH, HI \times HO, NO \times OM, \) etc. will form a geometric progression, the ratio of which will be \( AH/AG \).”

At this stage in his proof, Fermat makes two big “leaps.” First, he applies his interpretation of Euclid’s Proposition 35 (Book IX): Given a geometric progression the terms of which decrease indefinitely, the difference between two consecutive terms of this progression is to the smaller of them as the greater one is to the sum of the following terms. And second, he uses the process of “adequation” developed by both Archimedes and Diophantus, whereby one can basically “equate” a number and its approximation via a limiting process.

By Euclid’s Proposition, Fermat argued that the ratio \( GH \) (i.e. the difference of the first two terms \( AG \) and \( AH \)) to the smaller term \( AG \) equals the ratio of \( GE \times GH \) (the first rectangle) to the sum of all the other rectangles “in infinite number.” Using adequation and an earlier remark that the rectangles widths are very small, Fermat concluded that this latter “sum is the infinite figure bounded by \( HI \), the asymptote \( HR \), and the infinitely extended curve \( IND \).” That is, \( \frac{GH}{AG} = \frac{GE \times GH}{area\ DIHR} \).

4. Use this proportion to show that the rectangular area\( (AG \times EG) = area(DIHR) \). That is, the desired rectilineal area was in the diagram from the beginning!

Fermat then ended his proof with these words: If we add on both sides the parallelogram \( EG \times GH \), which, because of infinite subdivisions will vanish and will be reduced to nothing, we reach a conclusion…that for this kind of hyperbola the parallelogram \( AE \) is equivalent to the area bounded by the base \( EG \), the asymptote \( GR \), and the curve \( ED \) infinitely extended.

Fermat knew that his “uniform and general procedure” would work for the quadrature of all hyperbolas, except for one. The problem case was Appolonius’s “first” hyperbola, represented by \( xy = 1 \). In 1647 and 1661, this special case was resolved by Gregory of St. Vincent and Huygens respectively.

Extension and Reflection Questions:

Extension 1: Using the modern notation \( a, ar, ar^2, \ldots, ar^n, \ldots \) for an infinite decreasing geometrical series, Euclid’s claim is equivalent to what expression? Also, how does this “claim” and expression lead to the standard formula for the sum of such a series?

Extension 2: Using Fermat’s work, assume that the hyperbola’s equation is \( y = \frac{1}{x^2} \) and that \( AG \) is the unit length. Use modern integration techniques to confirm Fermat’s conclusion that “parallelogram \( AE \) is equivalent to the area bounded by the base \( EG \), the asymptote \( GR \), and the curve \( ED \) infinitely extended.” What if \( AE = a \) for \( a > 0 \)?
Extension 3: Fermat’s effort can be replicated in another manner as well, perhaps revealing some of the underlying dynamics lost in his somewhat cumbersome notation. Again, assume that the hyperbola’s equation is \( y = \frac{1}{x^2} \) that AG is the unit length, and the ratio \( \frac{AH}{AG} = p > 1 \). Partition the x-axis into the subintervals GH = [1,p), HO = [p,p^2), OM = [p^2,p^3), … and construct the ordinates EG, IH, NO,… as before. Then, rectangle EH has area \((1)(p-1)\), rectangle IO has area \(\frac{1}{p^2}(p^2 - p)\), rectangle NM has area \(\frac{1}{p^3}(p^3 - p^2)\), etc. First, for this set of subintervals, find the cumulative area of all of the rectangles. Second, explain how Fermat concluded that the desired area DEGR under the hyperbola is 1.

Extension 4: Fermat extended this same technique to the case of a parabola, such as \( y = x^n \) for \( n \) a positive integer. Basically, in modern notation, he applied an “inverse” method to divide the interval \( 0 \leq x < a \) using the points \( x_1 = a, x_2 = ar, x_3 = ar^2, \ldots, x_k = ar^{k-1}, \ldots \) where \( 0 < r < 1 \).

The corresponding ordinate values are \( y_1 = a^n, y_2 = a^n r^n, y_3 = a^{n-2} r^n, \ldots, y_k = a^{n-k} r^{(k-1)} \), … Complete Fermat’s reasoning:

- Show that the “infinite” sum of the areas of the circumscribed rectangles for a given \( r \) is \( S_r = \sum_{i=0}^{\infty} ar^i (1-r)(ar^i)^n \).
- Show \( S_r = \left[ \frac{1-r}{1-r^{n+1}} \right] a^{n+1} = \frac{a^{n+1}}{1+r+r^2+r^3+\ldots+r^n} \).
- Applying the idea of limits to \( S_r \) as \( r \to 1 \), show \( \int_0^a x^n dx = \frac{a^{n+1}}{n+1} \).

Eventually, Fermat was able to generalize his method for quadrature of the general case \( y = x^n \) (parabola or hyperbola) for all values of \( n \) except \( n = -1 \).

Open-ended Exploration: Investigate why Fermat’s approach breaks-down for the special case of the hyperbola represented by \( xy = 1 \) and AG being the unit length.