Greek Approximations of Square Roots

Historical Context:
- When: ca. 370 A.D.
- Where: Greece
- Who: Theon of Alexandria
- Mathematics focus: Demonstrate use of iteration to approximate square roots.

Suggested Readings:
- Theon of Alexandria: http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Theon.html
- The sexagesimal numeral system, as used by Theon: http://en.wikipedia.org/wiki/Sexagesimal
- NCTM’s *Historical Topics for the Mathematics Classroom* (1969): “Incommensurables and irrational numbers” (pp. 70-72) and “Radical symbol” (pp. 147-148)
- Key search words/phrases: Greek mathematics, square roots, iteration, approximation, Theon, sequence, sexagesimal, base sixty

Problem to Explore:
Find the square root of a positive number m.

Why This Problem is Important:
- Using Theon’s method, the Greeks’ iterative approach to extracting approximations to square roots is visual, intuitive, and powerful.
- Theon’s extraction of square roots is geometrical, yet is based on the binomial expansion in algebra.

Problem Solving Experiences:
Theon and his daughter Hypatia are considered the primary representatives of mathematical work being done in the last stages of Ancient Greece. In his *Commentary on Ptolemy’s Almagest*, Theon explains how to extract the square root of a sexagesimal number to the nearest second:

“When we seek a square-root, we take first the root of the nearest square-number. We then double this and divide with it the remainder reduced to minutes and subtract the square of the quotient, then we reduce the remainder to seconds and divide by twice the degrees and minutes (of the whole quotient). We thus obtain nearly the root of the quadratic.” (Gow, 1968, p. 56)

The meaning of this process needs to be explored and explained.
First, Theon’s algorithm is based on Postulate 4 (Book 2) in Euclid’s *Elements*, which claimed: “If a straight line is cut at random, then the square on the whole equals the sum of the squares on the segments plus twice the rectangle contained by the segments.” But, this is nothing more than a visual interpretation of the binomial expansion: \((a+b)^2 = a^2 + b^2 + 2ab\).

Theon first illustrated his process by finding \(\sqrt{144}\), even though he knew its root. His steps were as follows:

- Take a lesser square, say 100, whose root is 10
- Subtract 100 from 144, giving a difference of 44
- Multiply 10 by 2 to equal 20, because there are two rectangles
- Divide 44 by 20 to get 2 with a remainder of 4
- The 4 is the final small square, whose root is 2
- Thus, \(\sqrt{144} = 10 + 2 = 12\)

1. Draw the visual interpretation of Theon’s steps. Repeat his extraction process two more times, once using an initial “lesser square” of 81 and 64. What happens (i.e. what do you learn)?

Theon then illustrated his process by finding \(\sqrt{4500}\), which does not have an integral root. In sexagesimal notation, his steps for \(\sqrt{4500}\) were as follows:

- Start with a lesser square, say \(4489^0\), whose root is \(67^0\)
- Take this square with area \(4489^0\) and side \(67^0\) from the original square of area \(4500^0\), leaving a gnomon (i.e. l-shaped piece) with area \(4500^0 - 4489^0 = 11^0\).
- “Reduce” \(11^0\) to \(660'\) (i.e. shifting place value in base 60)
- Multiply \(67^0\) by 2 to equal \(134^0\), because there are two rectangles forming the arms of the gnomon
- Divide \(660'\) by \(134^0\) to get 4 with a remainder of 124'
- Dividing the area of \(536'\) into two rectangles, each with area \(4' \times 67^0 = 268'\).
- Unfortunately, the remainder of 124’ is greater than the desired square with area \(4' \times 4' = 16'\). Thus, inside our original square of area \(4500^0\) we have formed visually a square of area \(4497^056'16''\) and side length \(67^04'\) plus a new gnomon with remaining area of \(124' - 16'' = 7440'' - 16'' = 7424''\).
Repeat the process by multiplying $67^0 4'$ by 2 to equal $134^0 8'$ because there are two rectangles forming the arms of the gnomon.

Divide $7424''$ by $134^0 8'$ to get $55''$ with a remainder of $46''40''$.

Dividing the area of $7377''20''$ into two rectangles, each with area $55' \times 134^0 8' = 3688''40''$.

Theon then stopped the process, noting that “the remainder is $46''40''$, which is the square of which the side is nearly 55.”

2. Draw the visual interpretation of Theon’s steps for this more complicated root extraction. What is the numerical size of his error when he halts the process?

3. Repeat Theon’s extraction of $\sqrt{4500}$, but use our base ten system. What is the numerical size of his final error when he halts the process?

**Extension and Reflection Questions:**

**Extension 1:** The root $\sqrt{4500}$ was extracted in both base sixty and base ten, using the same number of iterations. Yet, the error terms differed, with the error in base 60 being much smaller. Explain why this is perhaps expected?

**Extension 2:** Mathematics historians often credit the Greeks with the discovery that the length of the diagonal of a unit square is not rational. That is, by the Pythagorean Theorem, the Greeks discovered that $\sqrt{2}$ was irrational. Though there are many different proofs of the irrationality of $\sqrt{2}$, the proof by contradiction is a classic. Find and study this proof (e.g. at [http://www.math.utah.edu/~pa/math/q1.html](http://www.math.utah.edu/~pa/math/q1.html)). Adapt the proof’s approach to prove that $\sqrt{3}$ is irrational. Also, what happens if you try to use the same approach to prove that $\sqrt{4}$ is irrational? (Note: The proof should fail.)

**Extension 3:** Plato credits Theodorus of Cyrene, a 5th century Greek mathematician, with the first proof of the irrationality of the square roots of 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15 and 17. Theodorus’ method of proof is not known, nor do we know why he stopped at 17. Some attribute the latter to an interesting geometrical technique that represents $\sqrt{n}$ for all positive integers:

- Draw an isosceles right triangle $A$ with side length 1
- Using the hypotenuse of triangle $A$, construct a perpendicular of length 1 to form another triangle $B$. 
- Continue this process using the hypotenuse of triangle B, construct a perpendicular of length 1 to form another triangle C, etc. What are the lengths of the hypotenuses of the triangles being formed? What does this construction have to do with Theodorus possibly stopping at $\sqrt{17}$?

Open-ended Exploration: Theon’s algorithm only extracted square roots for positive numbers. Can you make adjustments that also will extract cube roots of positive numbers? Nth roots of positive numbers?