Leibniz and Sums of Infinite Sequences

Solution Commentary:

Solution of Main Problems:

1. The series of odd integers is \(1 + 3 + 5 + 7 + 9 + 11 + 13 + 15\) can be represented visually as:

\[
\begin{array}{cccccccc}
0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 \\
1 & 3 & 5 & 7 & 9 & 11 & 13 & 15
\end{array}
\]

where each odd integer is the difference of the two perfect squares above it, such as \(3 = 4 - 1\), \(11 = 36 - 25\). Thus, the sum \(1 + 3 + 5 + 7 + 9 + 11 + 13 + 15\) = \((1 - 0) + (4 - 1) + (9 - 4) + (16 - 9) + (25 - 16) + (36 - 25) + (49 - 36) + (64 - 49)\) = \(64 - 0\) = \(64\). Notice that \(64 = 8^2\) and 15 is the 7th term in the sequence. Thus, when facing the sum \(1 + 3 + 5 + \ldots + 397\), we notice that 397 is the 199th odd integer, and thus \(1 + 3 + 5 + \ldots + 397 = (1 - 0) + (4 - 1) + \ldots + (199^2 - 198^2) = 198^2 - 0\). To generalize, applying Leibniz’s premise to the sum of \(n\)-terms, we have \(1 + 3 + 5 + \ldots + (2n-1) = (1-0) + (4-1) + \ldots + [n^2-(n-1)^2] = n^2 - 0^2\).

2. The proof can be done in multiple ways. One approach is to use the reverse sum idea. That is, if \(1 + 3 + 5 + \ldots + (2n-1) = S\), then \([1 + 3 + 5 + \ldots + (2n-1)] + [(2n-1) + \ldots + 5 + 3 + 1] = 2S.\) But, the left side of \(n\)-terms “collapses” into \(n(2n) = 2n^2\) and \(2n^2 = 2S\) implies that \(S = n^2\). A second approach is to use math induction. First, as the anchor, we have \(1 = 1^2\). Then, assuming \(1 + 3 + 5 + \ldots + (2n-1) = n^2\), we have \(1 + 3 + 5 + \ldots + (2n-1) + (2n+1) = n^2 + (2n+1) = (n+1)^2\). If the leading term is not \(1\), then the idea of a gnomon \([i.e. n^2 + (2n+1) = (n+1)^2]\) becomes useful. That is, for an arbitrary add integer \(m_i\), we have the identity \(m_i^2 = m_{i+1} - 2\), which can be verified by expansion of the squared terms. But, since \(m_i = m_{i+1} - 2\), substitute to get the revised identity \(m_i^2 = m_{i+1}^2 - 2\). Thus, for the sum of a general sequence of \(n\) consecutive odd integers, we have \(m_1 + m_2 + m_3 + \ldots + m_n = \left[\left(m_1^2 - \left(m_1 - 1\right)^2\right) + \left(m_2^2 - \left(m_2 - 1\right)^2\right) + \ldots + \left(m_n^2 - \left(m_n - 1\right)^2\right)\right]

= \left[\left(m_1^2 - m_0^2\right) - \left(m_1 - m_0\right)\right] or \left[\left(m_n^2 - m_0^2\right) - \left(m_n - m_0\right)\right].\)

3. From the first ten partial sums, it is difficult to make conclusions about the possible convergence of this sum to a finite limit:

<table>
<thead>
<tr>
<th>(S_1)</th>
<th>(S_2)</th>
<th>(S_3)</th>
<th>(S_4)</th>
<th>(S_5)</th>
<th>(S_6)</th>
<th>(S_7)</th>
<th>(S_8)</th>
<th>(S_9)</th>
<th>(S_{10})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.3333</td>
<td>1.5</td>
<td>1.6</td>
<td>1.6667</td>
<td>1.7143</td>
<td>1.75</td>
<td>1.7778</td>
<td>1.8</td>
<td>1.8182</td>
</tr>
</tbody>
</table>
4. By definition, \( t_n = 1+2+3+\ldots+n \), which implies that \( 2t_n = (1+2+3+\ldots+n) + (n+\ldots+3+2+1) = n(n+1) \). Thus, \( t_n = \frac{n(n+1)}{2} \). This relationship can be shown visually by dissecting a rectangular array of \( n(n+1) \) dots. For example, when \( n = 4 \) and \( t_4 = 1+2+3+4 = 10 \), we have \( t_4 = \frac{4(4+1)}{2} \):

![Diagram](image)

5. Observing the pattern \( \frac{1}{(1)(2)} = \frac{1}{1} - \frac{1}{2} \), \( \frac{1}{(2)(3)} = \frac{1}{2} - \frac{1}{3} \), \( \frac{1}{(3)(4)} = \frac{1}{3} - \frac{1}{4} \), the general term becomes \( \frac{1}{(n)(n+1)} = \frac{1}{n} - \frac{1}{n+1} \) which is true via a common denominator.

The, by Leibniz’s reasoning, we have

\[
2 \left[ \frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + \ldots + \frac{1}{(n)(n+1)} \right] = 2 \left[ \frac{1}{1} - \frac{1}{n+1} \right].
\]

Refocusing on sum \( S \) of the infinite series \( 2 \left[ \frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \ldots + \frac{1}{(n)(n+1)} + \ldots \right] \) where partial sum \( s_n = 2 \left( \frac{1}{1} - \frac{1}{n+1} \right) \), we have \( S = \lim_{n \to \infty} s_n = \lim_{n \to \infty} 2 \left( \frac{1}{1} - \frac{1}{n+1} \right) = 2 \). But, this is also the sum of the reciprocals of the triangular numbers, or \( \frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \ldots = 2 \).

**Extension 1**: Rewrite the geometric series \( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots \) as the telescoping series

\[
\left( \frac{1}{2} - \frac{1}{4} \right) + \left( \frac{1}{4} - \frac{1}{8} \right) + \left( \frac{1}{8} - \frac{1}{16} \right) + \ldots \].
\]

Notice that these consecutive subtraction of lengths are found in the second diagram, while the sum \( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots \) can be represented by visually summing the lengths right-to-left. Also, by this visual process,
it becomes apparent that since the “subtracted length” representing \( \frac{1}{n} \) collapses to 0, the sum \( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ... = 1 \).

**Extension 2:** The geometric sequence \( \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + ... \) can be represented as:

![Diagram of geometric sequence](image)

The associated telescoping series is \( \left(1 - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{16}\right) + \left(\frac{1}{16} - \frac{1}{64}\right) + \left(\frac{1}{64} - \frac{1}{256}\right) + ... \).

Reading the diagram right-to-left, the “new” infinite series \( \frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \frac{3}{256} + ... = 1 \).

Dividing both sides of this equation by 3, we conclude that \( \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + ... = \frac{1}{3} \).

Similarly, geometric sequence \( \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + ... \) can be represented as:

![Diagram of geometric sequence](image)

The associated telescoping series is \( \left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{27}\right) + \left(\frac{1}{27} - \frac{1}{81}\right) + ... \).

Reading the diagram right-to-left, the “new” infinite series \( \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + ... = 1 \).

Dividing both sides of this equation by 2, we conclude that \( \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + ... = \frac{1}{2} \).

**Extension 3:** Given \( \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^4} + ... \), The associated telescoping series is

\[
\left(1 - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n^2}\right) + \left(\frac{1}{n^2} - \frac{1}{n^3}\right) + \left(\frac{1}{n^3} - \frac{1}{n^4}\right) + ... .
\]

Without constructing a diagram that can be read right-to-left, the telescoping series becomes \( \frac{n-1}{n} + \frac{n-1}{n^2} + \frac{n-1}{n^3} + \frac{n-1}{n^4} + ... = 1 \), both visually and algebraically. Dividing both sides of this equation by \((n-1)\), we conclude that \( \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^4} + ... = \frac{1}{n-1} \).
Extension 4: Since \( \frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \ldots = 2 \left[ \frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + \ldots + \frac{1}{(n)(n+1)} \right] \), focus on the series \( \frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + \ldots + \frac{1}{(n)(n+1)} + \ldots = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots \).

Using \( AK = 1, AB = \frac{1}{2}, AC = \frac{1}{3}, AD = \frac{1}{4}, \ldots \), the latter series can be represented as:

The associated telescoping series is \( \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \ldots + \left( \frac{1}{n} - \frac{1}{n+1} \right) \).

Reading the diagram right-to-left, the “new” infinite series \( \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots = 1 \).

Multiplying both sides of this equation by 2, we conclude that \( \frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \ldots = 2 \).

Extension 5: The difficulty is that series \( A \) is the harmonic series, which is divergent. Thus, it is not appropriate to include it as a “defined” sum in an algebraic expression, which is later “casually” cancelled. For example, if such was allowed, we could justify weird conclusions such as \( 10 = \pi \); start with the equality \( \infty + 10 = \infty + \pi \) and then cancel the \( \infty \)-term by simple subtraction. To rectify the situation, Hofmann (1974) suggests that the focus be on combining the two parts as finite partial series, where

\[
A_n = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{n} \quad \text{and} \quad B_n = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \ldots + \frac{1}{n(n+1)}.
\]

Then, using the relationship \( \frac{1}{(n)(n+1)} = \frac{1}{n} - \frac{1}{n+1} \) and the same arithmetic process but now involving a finite number of terms, \( A_n + \frac{1}{2} B_n = 1 + 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \frac{1}{n+1} = 1 + A_n - \frac{1}{n+1} \) or

\[
B_n = 2 - \frac{2}{n+1}.
\]

Finally, \( \lim_{n \to \infty} B_n = 2 \).

Open-Ended Exploration: This investigation encourages students to link their current knowledge with the methods of Leibniz. The required level of sophistication is knowledge of calculus-level sequences and series. If students approach this investigation with an open mind, several surprises await them, plus their knowledge will be deepened.
When confronting Problems #1 and #2, it may be necessary to review the idea of a gnomon. For example, in this array connecting 4² and 5², the gnomon is 5 = 2(4)+1.

And from such an array, we can make the generalization \( n^2 + (2n + 1) = (n+1)^2 \), which also is the expansion of a trinomial. Also, be sure that students check their final identity using a known sequence of consecutive odds, such as 5 + 7 + 9 + 11.

Hofmann (1974, pp. 12-20) is the source for most of the content in this problem-solving experience, especially in the Extension Problems. As documentation, Leibniz described the library episode and the “heavy-handed writing style” in letters to Jakob Bernoulli in 1703 and Jean Gallois in 1672, respectively.

In Extension Problem #4, emphasize that Leibniz was using his technique for summing infinite geometric series to a situation involving the sum of a non-geometric series. So, then a good question to pose relates to extending this technique to any infinite series, as to whether or not it can show convergence and divergence. One key assumption is that the terms are limiting to 0. Another question to ask what happens when one tries this technique on the divergent harmonic series.

Extension Problem #5 is important because it illustrates the “loose” way that 17th and 18th century mathematicians (e.g. Leibniz, Euler, and Newton) misused divergent series to obtain correct mathematical results. Unfortunately, the technique seems to re-invent itself in the mathematical efforts of many students today, thus the renewed need for caution. Thus, push students to determine the source of error, rather than giving it to them too quickly.

As a good source of writing projects, students can explore any of the following ideas relative to Leibniz and his exploration of infinite series:

- Develop an annotated time-line that documents the history of the analysis of infinite series. For example, some key results are paradoxical struggles with an infinite number of terms or Leibniz’s theorem that an alternating series is convergent if the absolute value of the terms decrease monotonically to zero. Hairer & Wanner (1996) and Dönmez (2000) are two helpful resources for starting this project.
- Investigate Grandi’s series 1 – 1 + 1 – 1 + 1 – 1 + ..., named after the Italian mathematician, philosopher, and priest Guido Grandi who gave a thorough
analysis of the series in 1703. With a discussion of the Cesàro sum, the Abel
sum, and the Borel sum, include the techniques of dilution, separation of
scales, and related series. Some initial resources are Rhodes (1971), Kline
(1983), http://en.wikipedia.org/wiki/Grandi%27s_series,
http://en.wikipedia.org/wiki/Summation_of_Grandi%27s_series, and

- Investigate the Basel Problem, which is to find the sum of the infinite series
  comprised of reciprocals of perfect squares $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$. Include Euler’s
  solution and extensions. Sandifer (2007a, 2007b) and Dunham (1999) provide
great discussions of Euler’s solution approach.

- As an extension of the Basel Problem, investigate another series investigated
  by Euler, and published as Theorem 1 in his paper “Variae Observationes
  Circa Series Infinitas” (1737): the infinite series $\frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + \ldots$,
  which is the sum of all unit fractions whose denominators are perfect powers
  of integers minus unity; the series is not only convergent but has a sum of 1.
  Though attributing the theorem to Goldbach, Euler’s proof illustrates the
  misuse of divergent series to obtain correct results, a technique that Euler
  made popular and was used frequently by other 17th and 18th century
  mathematicians. Be forewarned that the proposed resolution of this Theorem
  and its proof requires knowledge of nonstandard analysis. Some starting
  resources are Paradis, Viader, & Bibiloni (2004), Kline (1983), Dunham
  (1999), and Sandifer (2007b).

- Leibniz’s approach involved a geometrical representation that helps determine
  both the convergence and the sum for a series. Investigate other geometrical
  representations that can be used to visualize and work with infinite series. For
  example, another example is Alexander Bogomolny’s fascinating construction:

![Geometrical representation](http://www.cut-the-knot.org/pythagoras/a_series.shtml)

which shows that the sum of the series $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \ldots = 1$ again. It can be

**Additional References:**


