Squaring of the Lunes

**Historical Context:**
- **When:** ca. 400 B.C.
- **Where:** Greek Island of Chios
- **Who:** Hippocrates
- **Mathematics focus:** Use geometrical ideas to transform a curve-sided figure into an equiareal rectangle or square.

**Suggested Readings:**
- NCTM’s *Historical Topics for the Mathematics Classroom* (1969): “Early Greek geometry” (pp. 172-174).
- Key search words/phrases: Hippocrates, Greek mathematics, quadrature, lunes, squaring a circle

**Problem to Explore:**
Use simple geometrical techniques to transform a lune into a rectangle or square that has the same area as the lune.

**Why This Problem is Important:**
- Extends Euclid’s quadrature sequence with polygons (i.e. transforming a polygon to a square with the same area) to the case of figures with curved sides.
- Solves geometrical problems involving parts of a circle, without using a value for pi.
- Illustrates a case where a mathematical technique basically led to a dead-end, as it was eventually proven that quadrature of a circle is impossible.
**Problem Solving Experiences:**

By definition, a “lune” is plane figure formed when two circular arcs intersect to form its boundary. It is necessary that the radii of the two arcs be different. In the real world, one experiences lunes in the form of lunar eclipses:

![Lunar Eclipse](image)

In fact, the word “lune” comes from the Latin word *luna* for “moon.”

Suppose you are given the following situation involving a lune and a right triangle:

![Diagram](image)

The lune(CEBD) is formed by the intersection of the arcs of a circle with center A and radius AB and a circle with center M (midpoint of hypotenuse BC) and radius MB.

1. Prove that area(lune CEBD) = area(triangle ABC), where it is okay to use the value $\pi$.
2. Prove that area(lune CEBD) = area(triangle ABC), where it not okay to use the value $\pi$.

One of Hippocrates’ first goals was to square a specific lune whose outer border was a semicircle, in contrast to the previous case of a quarter-circle. Heath () outlines Hippocrates’s plan of attack: *circumscribing a semicircle about an isosceles right-angled triangle and a segment of a circle similar to those cut off by the sides. Then, since the segment about the base is equal to the sum of those about the sides, it follows that, when the part of the triangle above the segment about the base is added to both alike, the lune will be equal to the triangle. Therefore the lune, having been proved equal to the triangle, can be squared.*
Try to replicate Hippocrates argument, by:

3. Prove that $\text{area(region(b))} = 2 \times \text{area(region(a))}$
4. Explain how this equality justifies that lune(AFBD) can be squared.

Hint: Hippocrates invoked a variation of what eventually became Euclid’s Proposition 19 (Book 6): *Similar shapes are to one another in the duplicate ratio of their corresponding sides*, where “duplicate ratio” means “ratio of their squares.”

As a final note, Hippocrates hope was that by achieving this result, he would be able to “square a circle.” However, that was eventually proven impossible.

**Extension and Reflection Questions:**

**Extension 1:** Inscribe a regular hexagon (label it ABCDEF) inside a circle with radius $MD = \frac{1}{2} AD$. Create a lune by constructing, exterior to the hexagon, a semicircle on segment $AB$ as diameter. Repeat this construction process five more times using the remaining sides of the hexagon as diameters.

Show, without using $\pi$, that the area(hexagon ABCDEF) = [areas of the six equal lunes] + [area of circle with diameter MD]. Hint: Make good use of Hippocrates variation of Euclid’s Proposition 19 (Book 6).
Extension 2: The above proof seems to imply that a circle can be squared. That is, \[\text{area of circle on radius MD} = \text{area(hexagon ABCDEF)} - \text{[areas of the six equal lunes]}\]. And, by the work of Hippocrates and Euclid, both the hexagon and the lunes are squarable, which implies the circle is squarable. What is wrong with this argument, since we now know such is impossible?

Open-ended Exploration: Extend this investigation to answer the question: How many constructible squarable lunes exist? First, Hippocrates found three types (isosceles triangle lune, isosceles trapezoid lune, and the concave pentagon lune). Then, in the mid-1700's two more types were discovered. Are there more…or do only five types of lunes exist?