Newton's Generalization of the Binomial Theorem

Historical Context:

- When: 1676
- Where: Cambridge, England
- Who: Isaac Newton
- Mathematics focus: Generalization of binomial coefficients and binomial expansions involving negative and fractional exponents.

Suggested Readings:

- Newton and his contributions to mathematics, physics, and philosophy:
  http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Newton.html
  or
  http://www.maths.tcd.ie/pub/HistMath/People/Newton/RouseBall/RB_Newton.html

- NCTM's *Historical Topics for the Mathematics Classroom* (1969): "Algebra in Europe (1200-1850)" (pp. 309-311), "The Binomial Theorem" (pp. 264-266), "Exponential notation" (pp. 327-331), and "Convergence" (pp. 432-434)
- I.B. Cohen's "Isaac Newton" (*Scientific American*, December 1955) or Chapter 5 in M. Kline's (ed.) *Mathematics in the Modern World* (1968), pp. 40-48
- Binomial expansion and Pascal's Triangle:
  http://mathforum.org/library/drmath/view/56381.html
- Key search words/phrases: Newton, British mathematics, Binomial expansion, Pascals triangle

Problem to Explore:

Investigate ways to generalize binomial expansions that involve fractional and negative exponents, such as $(p+q)^{1/2}$ and $(p+q)^{-3}$ respectively.

Why This Problem is Important:

- Generalizes Pascal's work with binomial coefficients and expansions far beyond the limits of his triangle.
- Develops the idea of infinite series as a basis for calculus and establishes the need for mathematical rigor.

Problem Solving Experiences:

When he introduced his Arithmetic Triangle in 1653, Pascal established key relationships within the expansion of binomial expressions. In its simplest form, for n a positive integer, $(p+q)^m = \sum_{k=0}^{m} \binom{m}{k} p^{m-k} q^k$ where $\binom{m}{k} = \frac{m!}{k!(m-k)!}$ and p, q may represent variables or numbers. Also, $\binom{m}{k}$ are the respective elements of base m of Pascal's Arithmetic Triangle (or row m of the modern configuration).

Newton belief in the "persistence of patterns" led to his first significant mathematics discovery, the generalization of the expansion of binomial expressions. For the most part, his discovery was accidental and was never formally proven by Newton.

On June 13, 1676, Newton wrote a letter to Leibniz, but it somehow was sent to Henry Oldenburg, secretary of the Royal Society. Introducing a fascinating but unusual visual pattern, Newton's generalization of the Binomial Theorem was:

The Extraction of Roots are much shortened by the Theorem

$$(P+PQ)^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + etc.$$

where P+PQ stands for a Quantity whose Root or Power or whose Root of a Power is to be found , P being the first term of that quantity, Q being the remaining terms divided by the first term and m/n the numerical Index of the powers of P + PQ…. Finally, in place of the terms that occur in the course of the work in the Quotient, I shall use A, B, C, D, etc. Thus A stands for the first term $P^{\frac{m}{n}}$ *; B for the second term (m/n)AQ; and so on.*

1. Given A = $P^{\frac{m}{n}}$, substitute sequentially to find expressions for B, C, and D. To maximize the "pattern effect" for generating expressions for E, F, etc., the denominator should contain only numbers.
2. The term P can be factored out on both sides since $(P+PQ)^{\frac{m}{n}} = P^{\frac{m}{n}}(1+Q)^{\frac{m}{n}}$. Thus, using your expressions in Problem #1, rewrite Newton's expression for $(1+Q)^{\frac{m}{n}}$. Also, substitute n=1 and m a positive integer, thereby obtaining the standard version of the Binomial Theorem.

Newton's theorem seems powerful given the relative lack of conditions on m and n. However, as an extension, it must preserve the results of whatever it is generalizing.

3. Letting Q = x and m/n = 3/1, show $(1+x)^3$ produces the expected expansion.
4. What is the expansion for $\sqrt{1-x} = (1-x)^{\frac{1}{2}}$, where Q = -x and m/n = 1/2?

Given the expansion for $\sqrt{1-x}=(1-x)^{\frac{1}{2}}$, Newton wrote that "extraction of roots are much shortened by this theorem." For example, to find the $\sqrt{7}$, Newton re-wrote the expression as $\sqrt{9\left(\frac{7}{9}\right)}=\sqrt{9\left(1-\frac{2}{9}\right)}=3\sqrt{1-\frac{2}{9}}$, and then used the expansion.

5. Complete Newton's approximation of $\sqrt{7}$ using the first six terms of the expansion. For each term, determine the error from the true value of $\sqrt{7}$.

Extension and Reflection Questions:

<u>Extension 1:</u> Newton's expression also held for cases where n = 1 and m was a negative integer. Let Q = x and m/n = (-3)/1 to determine the expansion for $(1+x)^{-3}$.

<u>Extension 2:</u> Determine the "correctness" of your expansion for $(1+x)^{-3}$ using these two approaches based on "finite" algebraic methods:

- Since $(1+x)^{-3}=\frac{1}{(1+x)^3}$, does 1 = $(1+x)^3$(your expansion….)?
- Since $(1+x)^{-3}=\frac{1}{(1+x)^3}=\frac{1}{1+3x+3x^2+x^3}$, use long polynomial division (i.e. $1+3x+3x^2+x^3\overline{)1}$) to see if the quotient is your expansion?

<u>Extension 3:</u> Though everything seems to "work" in the previous problems, unexpected difficulties can arise. For example, using Q = x and m/n = (-1)/1, the resultant expansion is $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots$. Now:

- If x = 1, we have $(1+1)^{-1}$ = ½ = 1 -1 + 1 – 1 + 1 – 1 + ….
- If x = 2, we have $(1+2)^{-1}$ = 1/3 = 1 – 2 + 4 – 8 + 16 – 32 + ….
- If x = -1, we have $(1-1)^{-1}$ = 1/0 = 1 + 1 + 1 + 1 + 1 + 1 +….
- If x = -2, we have $(1-2)^{-1}$ = -1 = 1 + 2 + 4 + 8 + 16 + ….

What is wrong, if anything?

<u>Open-ended Exploration:</u> Investigate Newton's use of his generalization of the Binomial Theorem to calculate a value of π correct to sixteen decimal places. And, why did Newton seemingly apologize for this calculation, relegating it to a "by the way" phenomena? In fact, because this work had been done during his personal retreat to Woolsthorpe during the Plague Years, Newton later wrote: "I am ashamed to tell you to how many figures I carried these calculations, having not other business at the time."