Pascal and His Arithmetical Triangle

Historical Context:
- When: 1653
- Where: Paris, France
- Who: Blaise Pascal
- Mathematics focus: Investigation of the structure of the Arithmetical Triangle, plus some of its properties and applications.

Suggested Readings:
- Pascal and his contributions to mathematics, physics, and philosophy: [http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Pascal.html](http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Pascal.html)
or
  [http://www.maths.tcd.ie/pub/HistMath/People/Pascal/RouseBall/RB_Pascal.html](http://www.maths.tcd.ie/pub/HistMath/People/Pascal/RouseBall/RB_Pascal.html)
- NCTM's *Historical Topics for the Mathematics Classroom* (1969): “Pascal's triangle” (pp. 156-157)
- Key search words/phrases: Pascal, French mathematics, Arithmetical triangle, geometry, binomial coefficients, combinations, figurate numbers

Problem to Explore:
Investigate the original structure of the Arithmetical Triangle and its connection to combinatorics.

Why This Problem is Important:
- Introduces a triangular arrangement of numbers that involves recursion, mathematics induction, and combinatoric relationships.
- It is a powerful connection to many mathematical areas—probability, algebra, combinatorics, fractal geometry, number theory, and recreational patterns.

Problem Solving Experiences:
Though known in China as early as the mid-1200’s, Pascal’s special triangle was the focus of his *Treatise on the Arithmetical Triangle* (1653). Pascal’s intent was to provide a convenient tabular presentation for binomial coefficients, which were connected to his proposed solution of the “division of stakes” problem. (Devlin, 2008)

The format of his arithmetical triangle (A.T.) differs visually from today’s standard:

Note that his notation was limited because of no access to subscripts. Thus, specific squares were denoted by arbitrary Latin and Greek letters. In turn, Pascal provided a rule for constructing the array:

*The number in the first cell by the right angle is arbitrary, but when this has been chosen, all the rest are determined. For that reason this number is called the “generator” of the triangle. Every other number is then determined according to the following rule: the number in each cell is equal to that in the preceding cell of the vertical column, increased by that in the preceding cell of the horizontal row. Thus cell F, that is, the number in cell F, is equal to cell C increased by cell E, and similarly for the rest.*

In modern notation, Pascal is claiming:
- Cell: \( p(m,n) = \) the number in \( m \)-th row and \( n \)-th column, i.e. \( p(3,4) = 10 \)
- Recursion rule: \( p(m,n) = p(m-1,n) + p(m,n-1) \)
- Generator: \( p(1,1) = 1 \)
- Unmentioned assumption: \( p(0,n) = p(m,0) = 0 \)

Based on observations of an Arithmetical Triangle with generator 1, Pascal established nineteen “consequences.” Though only a few of these will be discussed, the truth values for all of them can be established for generators other than 1 as well.

**Consequence 1:** The cells in the first row and those in the first column are equal to the generating number.
1. Rewrite this Theorem using modern notation, then indicate how you would prove it.

Consequence 2: Each cell is equal to the sum of the cells in preceding row, from the one in the same column to the first.

2. Rewrite this Theorem using modern summation notation, then indicate how you would prove it.

Consequence 3: Each cell is equal to the sum of the cells in preceding column, from the one in the same row to the first.

3. Rewrite this Theorem using modern summation notation.

Consequence 5: Each cell is equal to its reciprocal cell.

4. Rewrite this Theorem using modern summation notation.

After establishing basic properties of the Arithmetical Triangle, Pascal applies the Triangle to the theory of combinatorics. First, Pascal identifies the columns and rows as being figurate numbers (explored in Extension Problem #1). Then, Pascal connects the cells in his Arithmetical Triangle to combinatorial numbers. To prove his next Proposition, Pascal first needed to establish a Lemma.

Lemma 4: If there are four arbitrary numbers, of which the first is arbitrary, the second one greater than the first, the third arbitrary but not smaller than the second, and the fourth one greater than the third, then the number of combinations of the first in the third, increased by the number of combinations of the second in the third, is equal to the number of combinations of the second in the fourth.

In this Lemma, Pascal introduces the idea of combinatorics. Interpret the “number of combinations of j in k” the same as the “number of ways to pick j things from k things.” In modern notation, \( C_j^k = \binom{k}{j} = \frac{k!}{j!(k-j)!} \).

5. Rewrite the Lemma using modern summation notation, then prove it.

In stating his major Proposition, Pascal uses some new terminology. Again, in modern notation, Pascal would be using these ideas:

- A triangle of order N triangle is formed by its “base” of cells \( p(1,N) \leftrightarrow p(N,1) \)
- Cell \( p(m,n) \) is part of the base of an order N triangle if \( m + n - 1 = N \)
- The \( m \)-th-row of an order N triangle has \( N - m + 1 \) cells.

6. Determine a triangle of order 5, and interpret each of the above three ideas. How many cells are included in an order 5 triangle?
Pascal’s Proposition 1: In every Arithmetical Triangle, the sum of the cells in any row equals the number of combinations of row number in the order of triangle.

7. Rewrite the Proposition using modern summation notation and interpret \( \binom{5}{3} \) within this context for an order 5 triangle.

8. The value \( \binom{N}{m} \) is found in what cell of the Arithmetic Triangle? Check your claim by determining what cells contain the respective values of \( \binom{5}{1} \), \( \binom{5}{2} \), \( \binom{5}{3} \), \( \binom{5}{4} \), and \( \binom{5}{5} \).

Armed with this Proposition, Pascal applied his Arithmetical Triangle in a lengthy discussion of the Problem of Points. Finally, almost as an afterthought, Pascal stated that binomial expansions can be produced quickly using the corresponding cells in the “base” of an order \( N+1 \) triangle:

\[
(a + b)^N = a^N + \binom{N}{1} a^{N-1} b^1 + \binom{N}{2} a^{N-2} b^2 + \ldots + \binom{N}{N-1} a^1 b^{N-1} + b^N
\]

Extension and Reflection Questions:

Extension 1: Figurate numbers are based on arrangements of equally-spaced points to form regular geometrical shapes. In two-dimensions, the triangular numbers are 1, 3, 6, 10, 15,... This diagram shows the geometry involved, plus the consecutive rows being added reveals the relationship of the natural numbers to triangular numbers: 1, 3 = 1+2, 6 = 1+2+3, 10 = 1+2+3+4, etc.

In Pascal’s Arithmetical Triangle, the third column lists all of the triangular numbers. Determine how the subsequent columns (e.g. 1, 4, 10, 20,...) are also figurate numbers.
Extension 2: Using Problem #8 and Pascal’s Proposition, why is it seemingly impossible for Pascal to obtain a value for \( \binom{5}{0} \)? Yet, why is it reasonable that Pascal, if asked, would conclude that \( \binom{5}{0} = 1 \)?

Extension 3: Consider two more of Pascal’s Consequences, as expressed in modern notation:

- **Consequence 8:** \( \sum_{m,n=1}^{N} \binom{m+n}{m} = 2^{N-1} \) when \( m+n = N+1 \)
- **Consequence 9:** \( 1 + 2 + 2^2 + ... + 2^{N-1} = 2^N - 1 \)

In your own words, explain what these two Consequences are claiming relative to “bases” of an order N Arithmetical Triangle. Illustrate this meaning for an order 5 triangle. And, use mathematics induction to prove the Consequences.

Extension 4: Use mathematics induction to prove Pascal’s Proposition 1. **Hint:** Lemma 4 will prove useful as well.

Open-ended Exploration: Investigate the extension of Pascal’s “triangle” into N-dimensions, relating the cells to the multinomial coefficients of \( (a_1+a_2+...+a_n)^N \). For example, Pascal’s tetrahedron will provide the trinomial coefficients of \( (a_1+a_2+a_3)^N \). Also, is there an equivalent formula for finding the value of a specific cell using combinations?