The Problem of Points

Historical Context:

- When: 1494-1654
- Where: Italy and France
- Who: Luca Pacioli, Girolamo Cardan, Niccolo Tartaglia, Blaise Pascal and Pierre de Fermat
- Mathematics focus: Investigation of the idea of expected value in a fair game situation.

Suggested Readings:

- Luca Pacioli and his contributions to mathematics:
  [http://www-history.mcs.st-and.ac.uk/Biographies/Pacioli.html](http://www-history.mcs.st-and.ac.uk/Biographies/Pacioli.html)

- Girolamo Cardano and his contributions to mathematics:
  [http://www-history.mcs.st-andrews.ac.uk/Biographies/Cardan.html](http://www-history.mcs.st-andrews.ac.uk/Biographies/Cardan.html)

- Niccolo Tartaglia and his contributions to mathematics:
  [http://www-history.mcs.st-andrews.ac.uk/Biographies/Tartaglia.html](http://www-history.mcs.st-andrews.ac.uk/Biographies/Tartaglia.html)

- Pascal and his contributions to mathematics:
  [http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Pascal.html](http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Pascal.html)

- Fermat and his contributions to mathematics:
Problem to Explore:
Investigate the appropriate division of stakes in a fair game.

Why This Problem is Important:
- Often referred to as the “catalyst” for the start of probability theory, especially the idea of expected value.
- Reveals how real-world situations prompt the creation of new mathematics.

Problem Solving Experiences:

In his *Summa de arithmetica, geometria, proportioni et proportionalita* in 1494, Luca Pacioli, an Italian mathematician and Franciscan friar, posed a problem: *A team plays ball in such a way that a total of 60 points is required to win the game, and each goal counts 10 points....By some incident they cannot finish the game and one side has 50 points and the other 30. What share of the prize money belongs to each side?*

1. Assume that P is the team ahead with goals and Q is the other team. Respond (with reasoning) to the “fairness” of these three proposed solutions:
   - Give P all of the prize money as it is ahead
   - Divide the prize money equally, because the game was not completed
   - Since 8 goals have been scored, give P 5/8 of the prize money and Q the remaining 3/8 (Pacioli’s proposed solution)

In his *Practica arithmetica* in 1539, Girolamo Cardano, an Italian mathematician and avid gambler, considered the problem and discounted Pacioli’s solution as containing an error that “even a child should recognize.” In turn, Cardano’s solution focused on the number of goals N needed to win the game rather than the number of goals already scored by P and Q. Then, the prize money should be allocated using the general ratio \([1+2+3+...+(N-Q)]:[1+2+3+...+(N-P)].\)

2. Using Cardano’s general proposal, what share of the prize money would each team receive? Is this division fair?
In his *Generale Trattato* in 1556, Tartaglia, an Italian mathematician and rival of Cardano, also discounted Pacioli’s solution as “neither reasonable nor good.” His proposed solution focused on the difference in goals \((P-Q)\) of the two teams relative to the number of goals \(N\) needed to win the game. Then, he argued that \(P\) should get 
\[
\frac{1}{2} + \frac{P-Q}{2N}
\]
of the prize money and \(Q\) the remaining 
\[
\frac{1}{2} - \frac{P-Q}{2N}.
\]

3. Using Tartaglia’s general proposal, what share of the prize money would each team receive? Is this division fair?

In his *Practica d'arithmeticca e geometria* in 1603, Italian mathematician Lorenzo Forestani’s proposed an alternate solution to the problem. Stating that the key is the ratio of the number of games a team has won to the number of games played, he first gave each team their respective portion of their contributed stakes and then divided the remainder of the combined money prize evenly between the two teams.

4. Using Forestani’s proposal, what share of the prize money would each team receive? Is this division fair?

In 1654, Antoine Gombard, the Chevalier de Méré, was not only a French aristocrat and classical scholar, but also a compulsive gambler. Frustrated by his numerous losses, he posed a question quite similar to Pacioli’s original problem. Unfortunately, all copies of the problem have been lost, but mathematics historians have tried to reconstruct it: *Two players play a game of three points and each player has staked 32 pistoles. How should the sum be divided if they break off at any stage?* (David, 1962, p. 85)

Previous “solutions” to these “points” problems seemed incorrect, perhaps because the players had to receive proportional amounts reflecting their respective chances of winning the series without a premature termination. Thus, the problem greatly confused de Méré, perhaps due to his modest mathematical skills and keen empirical observations. Eventually, the frustrated de Méré shared the problem with Pascal and other mathematicians. Intrigued by the problem, Pascal offered possible solution(s), while also discussing the problem in multiple correspondences with Fermat. In the end, both Pascal and Fermat solved the problem, but using different reasoning.

First, Pascal reasoned that de Méré’s situation involved three cases: \((2,1)\), \((2,0)\), and \((1,0)\), where \((P,Q)\) represents one player having \(P\) points and the other \(Q\) points. In a letter to Fermat (July 29, 1654), Pascal wrote:

*Suppose that the first player has gained 2 points and the second player 1 point. They now have to play for a point on this condition, that if the first player wins he takes all the money which is at stake, namely 64 pistoles, and if the second player wins each player has 2 points, so they are on terms of equality, and if they leave off playing each ought to take 32 pistoles. Thus, if the first player wins, 64 pistoles belong to him, and if he loses, 32 pistoles belong to him. If then the players do not wish to play this game, but to separate without playing it, the first player would say to the second: “I am certain of 32 pistoles even if I lose this game, and as for the other 32 pistoles perhaps I shall have*
them and perhaps you shall have them; the chances are equal. Let us divide these pistoles equally and give me also the 32 pistoles of which I am certain.” Thus the first player will have 48 pistoles and the second 16.

5. Given Pascal’s analysis for the (2,1) case, conjecture how Pascal reasoned the stakes should be divided for both the (2,0) and the (1,0) cases? **Hint:** Think recursion!

6. Given the same situations of (2,1), (2,0), and (1,0), what should be the division of stakes according to Pacioli? Cardano? Tartaglia? Forestani? Do any of their amounts agree with Pascal’s?

In direct contrast to Pascal’s recursive argument and use of weighted averages, Fermat offered a different approach in his response letter (August 24, 1654). First, for the case (1,0), Fermat argued that at most 2+3-1 = 4 more games would be needed to declare a winner, as the first player needed 2 more points and the second player needed 3 more points. He then listed out all of the possible four game situations, and determined the division of stakes as proportional to the “ratio of wins” for each player.

7. Continuing with the case (1,0), list all of the possible outcomes for four games and determine the division of stakes. Then, apply Fermat’s strategy to the cases (2,1) and (2,0). Do Fermat’s amounts agree with Pascal’s?

8. Gilles de Roberval, a French mathematician and peer of both Pascal and Fermat, objected strongly to Fermat’s approach of listing outcomes. Any idea as to the basis of his objections?

Finally, in 1654, Pascal included another solution to the “points” problem in his *Treatise on the Arithmetical Triangle*. In modern form, his triangle looks like this:

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1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
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and a row, such as 1-3-3-1, forms a “base” for the triangle of numbers above it. Applying his triangle to the “points” problem, Pascal wrote:

*Let there be taken in the triangle the base which has as many cells as the two players together lack games; then let there be taken in this base as many contiguous cells beginning with the first as the first player lacks games, and let the sum of their numbers be taken. Therefore, there remain as many cells as the second player lacks games. Again let the sum of their numbers be taken. These sums are to each other as the odds of the players reciprocally…expressed by the fraction.*

9. Use Pascal’s Arithmetical triangle to solve de Méré’s problem of “points.”
Extension and Reflection Questions:

Extension 1: Use the three techniques of Pascal and Fermat (i.e. weighted averages, listing outcomes, and Arithmetical Triangle) to solve Pacioli’s problem of “points. Which method seems to be the best?

Extension 2: In 1654, Antoine Gombard, the Chevalier de Méré, asked Pascal a second question along with the problem of points. On the simplest level, de Méré would bet that he would get at least one ace (i.e. a 1) in four rolls of a dice. On a more advanced level, de Méré would bet that he would get at least one double ace (i.e. 1-1) in twenty-four rolls of a pair of dice. Using his limited mathematical knowledge, de Méré reasoned as follows:

- In one roll of a die, the chances of getting one ace is \( \frac{1}{6} \). Thus, in 4 rolls, the probability of getting at least one ace is \( 1 - \left(1 - \frac{1}{6}\right)^4 = \frac{2}{3} \).

- In turn, extending this same reasoning to the second game, in one roll of a pair of dice, the chances of getting at double ace is \( \frac{1}{36} \). Thus, in 24 rolls, the probability of getting at least one double ace is \( 1 - \left(1 - \frac{1}{36}\right)^{24} = \frac{2}{3} \).

Open-ended Exploration: Investigate the extension of the problem of points into the case involving three players or teams. Both Pascal and Fermat offered solutions for this extension. Generalize the solution to the problem of points for the case of n players or teams? What if the players do not have equal probabilities of winning?