Three Famous Problems of Antiquity

**Historical Context:**
- When: ca. 300 B.C.
- Where: Greece
- Who: Greek mathematicians
- Mathematics focus: Using construction tools to solve three problems: squaring of a circle, trisecting an angle, and duplicating the cube

**Suggested Readings:**
- Three problems of antiquity (or is it four):  
  2. [http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Squaring_the_circle.html](http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Squaring_the_circle.html)
  3. [http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Doubling_the_cube.html](http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Doubling_the_cube.html)
  4. [http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Trisecting_an_angle.html](http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Trisecting_an_angle.html)
- Greek geometric constructions  
- NCTM’s *Historical Topics for the Mathematics Classroom* (1969): “Compass and straightedge constructions” (pp. 192-196), “Duplication of the cube” (pp. 197-198), “The trisection problem” (pp. 199-201), and “The quadrature of the circle” (pp. 201-204).
- Key search words/phrases: Problems of antiquity, quadrature, squaring circle, trisecting angle, duplicating cube, Delian problem, impossible

**Problem to Explore:**
Using construction tools to solve three problems of antiquity: squaring a circle, trisecting an angle, and duplicating the cube.

**Why This Problem is Important:**
• Illustrates Greek fascination with theoretical side of geometry versus its practical side.
• These problems led to the creation of significant mathematics, being connected to proofs that \( \pi \) is transcendental and the application of Galois Theory.
• Illustrates how a problem can be proven impossible mathematically, yet individuals persist in trying to solve that problem.

**Problem Solving Experiences:**

The assumption is that you are aware of the Greeks’ three problems of antiquity:

- Squaring or quadrature of the circle
- Duplication of the cube (or Delian problem)
- Trisection of an arbitrary angle

Certainly all three of these problems are solvable. For example, inscribe a square inside a circle; now expanding the square continuously until the circle is inscribed inside the square. At some point in this process, the square and the circle had equal areas. The key is that the problems are unsolvable under the Greeks’ restriction that a solution had to be accomplished using only an unmarked straightedge and a compass.

Since the problems are impossible, our focus will be on investigating several “close” attempts. First, a construction that claimed to square a circle by deriving a length equal to \( \pi \). The argument was that if you could construct a segment of length \( \pi \), then you could construct a segment whose length was \( \sqrt{\pi} \). The latter segment could be used to construct a square of side length \( \sqrt{\pi} \) or area \( \pi \), the equiareal transformation of a circle with radius 1 or area \( \pi \).

Carefully consider the following set of steps, using the construction tools (or GSP):

- Draw a circle \( C \) with center \( O \) and radius \( OA \) (assumed to be unit length)
- Extend ray \( AO \) to intersect circle at point \( B \), forming diameter \( AB \).
- Draw line \( m \) tangent to circle \( C \) at point \( B \).
- Construct angle \( AOD \) of measure \( 30^\circ \) (or \( \frac{\pi}{6} \)) with point \( D \) being on line \( m \).
- On ray \( DA \), construct point \( E \) such that \( DE = 3(AO) \).
- Claim: The length of segment \( BE \) is \( \pi \) (or \( \frac{1}{2} \) the circumference of the circle).
1. Without using trigonometric functions, determine the numerical value of the length BE to four decimal places. Compare with the actual value to \( \pi \) to show that this is an approximate solution and not an exact solution.

The doubling of the cube problem has an interesting history: In 430 B.C., the Athenians supposedly were suffering from a plague of typhoid fever. The elders consulted the oracle at Delos, with the god Apollo replying that the plague would cease only if his cubical altar was doubled in size. The Athenians quickly built a new altar by doubling each side. Now irate, the god Apollo made the plague worse. In desperation, the Athenian elders turned to a group of mathematicians led by Plato, who referred them to the geometer Euclid. He tried to solve the doubling problem by using only a straight edge and compass, and failed.

2. What was the resultant size of the Athenians' altar that got Apollo so irate? And, what other historical errors are included in this story to make it apocryphal?

Consider this Lemma: Given a cubic equation \( p(x) \) with integer coefficients \( p(x) = a_3x^3 + a_2x^2 + a_1x + a_0 = 0 \), either one of the equation’s roots is rational or none of its roots are constructible. It will be assumed true here, but can be proven using extension fields in abstract algebra.

3. Use this Lemma to show that the duplication of the cube is impossible.

The Arabic work Book of Lemmas (circa 200 B.C.) contains a proposed solution to the trisection problem. Archimedes is credited with the problem (by Thabit ibn-Qurra), as well as much of the material in this book. His “mechanical” approach is very simple. Start with an arbitrary angle AOB and construct a circle C with center O and radii OA = OB. From point B, construct a secant line that that intersects the circle C at point E and the ray AO at point D, such that EF = OA. Construct radius OF interior to angle AOB such that OF is parallel to BD.

4. Prove that angle AOF = 1/3 (angle AOB). Also, does your proof hold if angle AOB is a right angle? An Obtuse angle?

5. What is the mistake in this construction that keeps it from overcoming the “impossible”?
**Extension and Reflection Questions:**

**Extension 1:** In the 1840s, a German author claimed he had “solved” the quadrature of the circle problem, and almost every page of his manuscript contained the words “thanks almighty God that he and no one else was selected to solve the problem phenomenal.” His solution was based on this diagram:

![Diagram of the circle problem](image)

His reasoning: The inside square involves 4 congruent triangles, while the outside square involves 8 of these same congruent triangles. Thus, the circle involves 6 of these congruent triangles. What is the value of $\pi$ determined by this author’s faulty construction?

**Extension 2:** An intuitive approach to trisecting a angle $\angle AOB$ is well-known and often proposed by students. Construct a circle $C$ with center $O$, intersecting the angle’s sides $OA$ and $OB$ at points $M$ and $N$ respectively. Trisect segment $MN$ to get points $P$ and $Q$. Construct $OP$ and $OQ$. The claim is that these interior segments trisect angle $\angle AOB$ as well.

![Diagram of angle trisection](image)

Do $OP$ and $OQ$ trisect angle $\angle AOB$? Does angle $MOP = angle POQ = angle QON$?

**Extension 3:** Consider the claim that a $60^\circ$ angle can be trisected. Few remember the esoteric trigonometric identity $\cos(3\Theta) = 4 \cos^3(\Theta) - 3 \cos(\Theta)$, but it proves helpful here. Letting $3\Theta = 60^\circ$, prove that the original trisection claim is false. **Hint:** The Rational Root Theorem also will useful in this proof.
Open-ended Exploration: Numerous mathematical curves and mechanical devices have been used to “solve” the these problems of antiquity, though they do not obey the restriction to use only of a straight edge and a compass. Investigate the role of curves such as Archimedes’ spiral, Nicomedes’ conchoid, Appollonius’ conchoid, Pascal’s limacon, and Carpus’ double-motion curve as well as mechanical devices such as the shoemaker’s knife (or tomahawk) and the carpenter’s square. Dudley (1996) is a great resource in this investigation. Origami and a cylinder wrapped by string also have been used to solve the trisection problem.

A sampling of the many published resources that may prove helpful:


Or, these internet resources will get you started:

- [http://www.math.lsu.edu/~verrill/origami/trisect/](http://www.math.lsu.edu/~verrill/origami/trisect/)
- [http://kahuna.merrimack.edu/~thull/omfiles/geoconst.html](http://kahuna.merrimack.edu/~thull/omfiles/geoconst.html)