MATH 124

Examination 2
November 4, 2008

Name Key

For a full credit give your answers as exact numbers, not decimal approximations.

1. We all know that $\pi \approx 3.14159$. A very popular rational approximation of π is $\frac{22}{7}$. This is the best approximation for π by a fraction with a two-digit denominator. The best rational approximation of π using a fraction with a three-digit denominator is $\frac{355}{113}$. Notice that $\pi < \frac{355}{113} < \frac{22}{7}$. Use an appropriate linear approximation of the function $f(x) = \sin x$ to find a rational approximation of the number $\sin\left(\frac{22}{7}\right)$. Your answer should look like $\sin\left(\frac{22}{7}\right) \approx \frac{a}{b}$, where a is an integer and b is a three-digit positive integer.

2. An object is launched vertically into the air and its distance from the ground (in feet) at any time t (in seconds) is given by $h(t) = 100(1 - e^{-t})$. The object is equipped with a remote operated cruise control device (i.e. we can fix the velocity of the object at any moment). Assume that we have fixed the velocity of the object at time $t = \ln 10$.

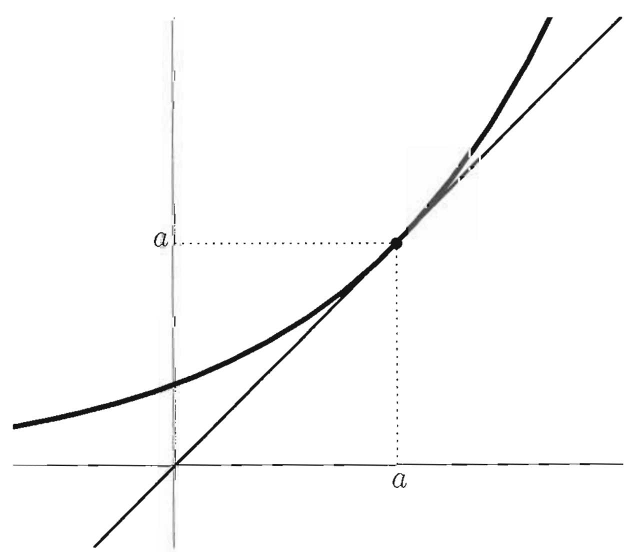
   (a) What is the velocity of the object at time $t = \ln 10$?

   (b) What is the height of the object at time $t = \ln 10$?

   (c) Assuming that the velocity have been fixed at the time $\ln 10$, give a formula for the height of the object for $t > \ln 10$.

   (d) When will the object reach the height of 100 ft?

3. The picture on the right shows an exponential function $y = f(x) = e^{kx}$ and its tangent line. The function and the tangent line touch at the point (a, a). The point $(0, 0)$ belongs to the tangent line. Determine the numbers k and a.



4. Consider the function $f(x) = x\,2^x$.

   (a) Calculate the first and the second derivative of f.

   Use the derivatives found in (4a) to identify the maximum intervals where:

   (b) (i) f is increasing; (ii) f is decreasing;

   (c) (i) f is concave up; (ii) f is concave down.

5. Differentiate each of the following functions:

   (A) $\cos(\sqrt{x})$ (B) $\sqrt{\cos(\sqrt{x})}$ (C) $\arctan\left(\frac{1}{x}\right)$ (D) $\sqrt{1+\sqrt{1+x^2}}$

For the full credit show all your work.

① $\sin x \approx \pi - x$ near π

$$\sin\frac{22}{7} \approx \pi - \frac{22}{7} \approx \frac{355}{113} - \frac{22}{7} =$$

$$= \frac{355 * 7 - 113 * 22}{113 * 7} = \boxed{\frac{-1}{791}}$$

② $h(t) = 100(1 - e^{-t})$, $h'(t) = 100e^{-t}$

ⓐ $h'(\ln 10) = 100\frac{1}{10} = 10$

ⓑ $h(\ln 10) = \boxed{90}$

ⓒ $l(t) = 10(t - \ln 10) + 90$

ⓓ $t = \boxed{1 + \ln 10}$

at vel. 10 it will take 1 sec to cover 10 ft.

③ $f(x) = e^{kx}$

$f'(x) = ke^{kx}$

$f'(a) = \boxed{ke^{ka} = 1}$

$f(a) = \boxed{e^{ka} = a}$

$e^{ka} = a \Rightarrow ke^{ka} = \boxed{ka = 1}$

$ka = 1 \Rightarrow \boxed{e^{1} = a}$, so $\boxed{\begin{array}{l} a = e \\ k = 1/e \end{array}}$

4

$$f(x) = x\,2^x$$

2

$$f'(x) = 2^x + x(\ln 2)\,2^x$$

$$\boxed{f'(x) = 2^x\left(1 + x(\ln 2)\right)}$$

$$f''(x) = 2^x \ln 2 + (\ln 2)2^x\left(1+x(\ln 2)\right)$$

$$\boxed{f''(x) = (\ln 2)2^x\left(2 + x(\ln 2)\right)}$$

(b(i)) $\boxed{f \text{ increasing}}$ where $f'(x) > 0$

$1 + x(\ln 2) > 0$ $\boxed{x > -\frac{1}{\ln 2}}$

(b(ii)) $\boxed{f \text{ decreasing}}$ where $f'(x) < 0$

$\boxed{x < -\frac{1}{\ln 2}}$

(c(i)) $\boxed{f \text{ concave up}}$ where $f''(x) > 0$

$2 + x(\ln 2) > 0$

$\boxed{x > -\frac{2}{\ln 2}}$

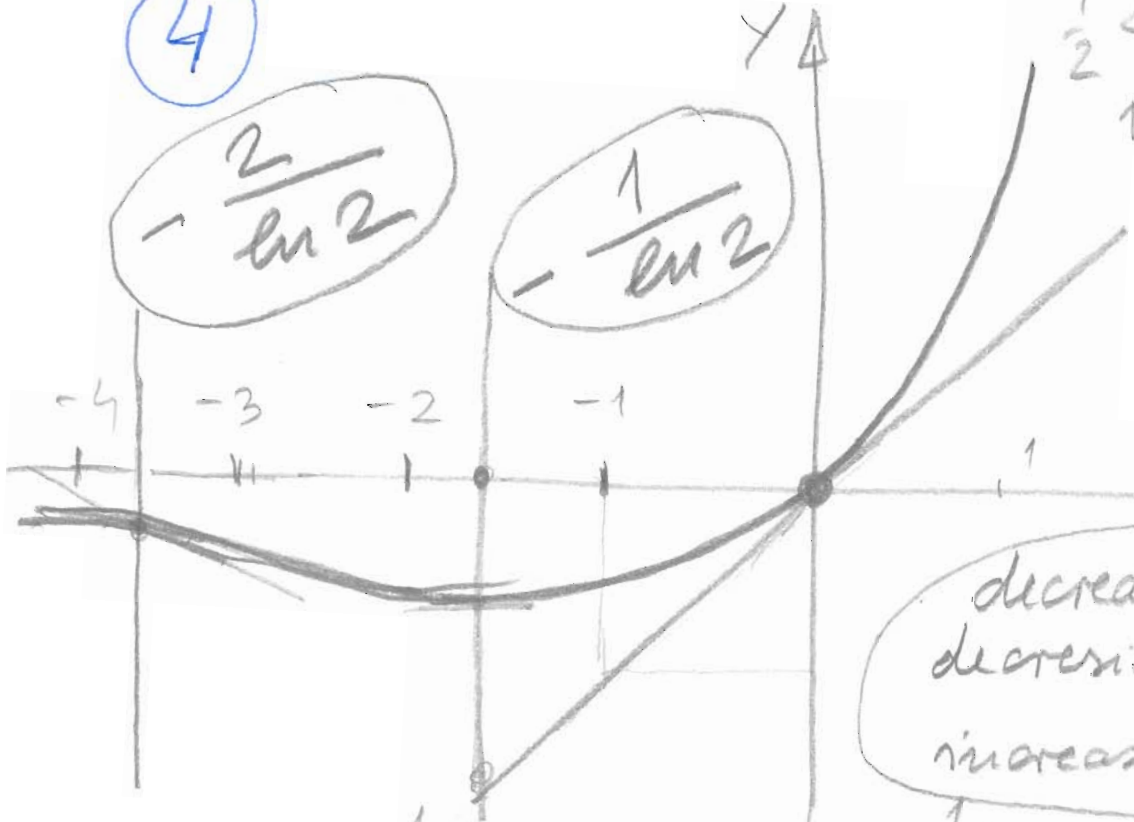
(c(ii)) $\boxed{f \text{ concave down}}$

$x < -\frac{2}{\ln 2}$

4

$\frac{1}{2} < \ln 2 < 1$

$1 < \frac{1}{\ln 2} < 2$ $\boxed{3}$



decreasing, concave d,
decresin, concave up
increasing, concave up

$$x = -\frac{1}{\ln 2} \neq -\frac{1}{\ln 2}\, 2^{-\frac{1}{\ln 2}} = -\frac{1}{\ln 2}\,\frac{1}{2}$$

$\frac{1}{\ln 2}$ ~~(ln 2)~~

$$x = -\frac{2}{\ln 2}\;)\quad -\frac{2}{\ln 2}$$

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5 (A) Use the chain rule $(\cos x)' = -\sin x$

$$\frac{d}{dx}(\cos\sqrt{x}) = \boxed{-\sin(\sqrt{x})\cdot\frac{1}{2\sqrt{x}}}$$

(B) Use the chain rule with $\sqrt{u}$ and the previous function

$$\frac{d}{dx}\left(\sqrt{\cos\sqrt{x}}\right) = \boxed{\frac{1}{2\sqrt{\cos\sqrt{x}}}\left(-\sin\sqrt{x}\right)\frac{1}{2\sqrt{x}}}$$

(C)

$$\frac{d}{dx}\left(\arctan\frac{1}{x}\right) = \frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right) = \boxed{-\frac{1}{1+x^2}}$$

(D) $\frac{d}{dx}\left(\sqrt{1+\underbrace{\sqrt{1+x^2}}}\right)$

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First $\frac{d}{dx}\left(1+\sqrt{1+x^2}\right) =$

$$= \frac{1}{2\sqrt{1+x^2}} \cdot 2x = \frac{x}{\sqrt{1+x^2}}$$

Now the chain rule ~~fo~~ with $\sqrt{u}$

$$\boxed{\frac{1}{2\sqrt{1+\sqrt{1+x^2}}} \cdot \frac{x}{\sqrt{1+x^2}}}$$