MATH 124

Final Examination
December 11, 2008

Name Key

GIVE EXPLANATIONS FOR YOUR ANSWERS.
There are eight problems. Each is worth 12.5 points.

1. (a) Suppose that a box with a square base and without a top has a surface area of one square unit. Find the dimensions that will maximize the volume.

   (b) A simple way to construct a box with a square base and without a top is to start with a square piece of cardboard. Assume it has one unit square area. The box is constructed by cutting out a small square from each of the four corners and bending up the sides. Find the dimensions that will maximize the volume of a box constructed in this way.

   (c) Compare the results of the previous two calculations and comment on what you observe.

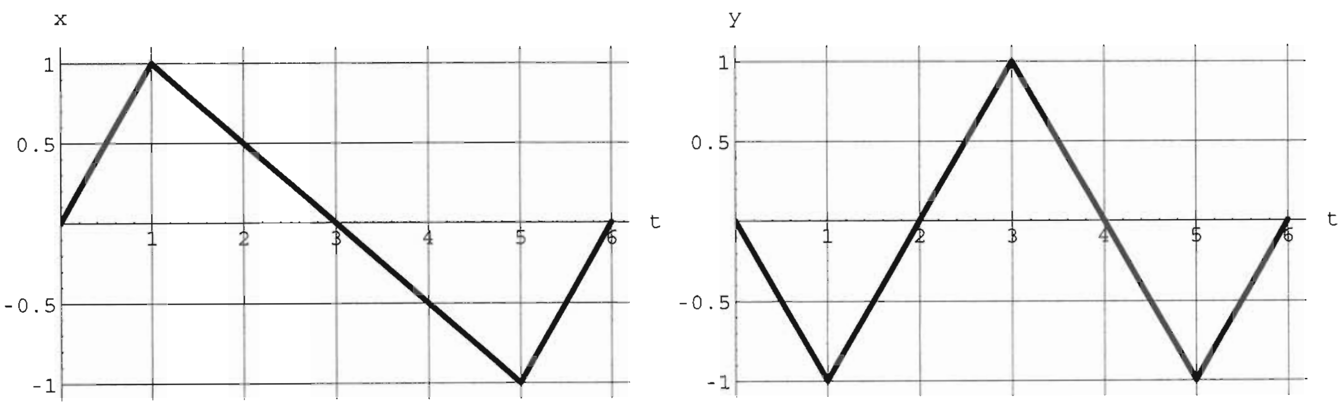
2. Consider the function $f(x) = \arctan(x) + \arctan\left(\frac{1}{x}\right), \quad x \neq 0$.

   (a) Graph the function f on your calculator. What do you observe? Based on the calculator graph, give an accurate graph of this function. (Clearly identify exact coordinates of the relevant points.)

   (b) What does (2a) imply about the derivative $f'(x)$?

   (c) Calculate $f'(x)$ by using special and general rules. Give a reasonably detailed explanation of which special and which general rules you use.

3. The figures below show the graphs of the functions $f(t)$ and $g(t)$. Describe the motion of the particle whose coordinates at time t are $x = f(t)$ and $y = g(t)$.



   Provide an accurate graph for the motion of the particle in the xy-plane with the direction of the motion indicated.

4. The position of a jogger during a two-hour exercise along a straight trail is given by the formula

$$p(t) = 3t^2e^{2-t}, \quad 0 \leq t \leq 2.$$

   The number $p(t)$ is the distance of the jogger from the trail-head in miles.

   (a) What was the jogger's average velocity (in miles per hour) during the exercise?

   (b) What was the jogger's maximum velocity?

   (c) Find the time interval starting at $t = 0$ during which the joggers's average velocity was the greatest?

OVER

5. Water is flowing into a conical reservoir at a rate of 4 m^3/hour. The reservoir is 3 m in radius at the top and 12 m deep.

   (a) At what rate is the depth increasing at the instant when the water is 8 m deep?

   (b) At what rate is the surface area of the top of the water increasing at the instant when the water is 8 m deep?

6. Consider the function $f(x) = \dfrac{x+1}{\sqrt{1+2x^2}}$.

   (a) Find the derivative of f. Show your work. Identify where the derivative is positive, where it is negative and find the critical point(s) of f.

   (b) Identify and calculate exactly the global maximum and the global minimum of f if they exist. Provide reasoning for your answers.

   (c) Find the inflection points of f. Identify the intervals where the function is concave up and concave down.

   (d) Draw a reasonably accurate graph of f. Indicate all the information found above on the graph.

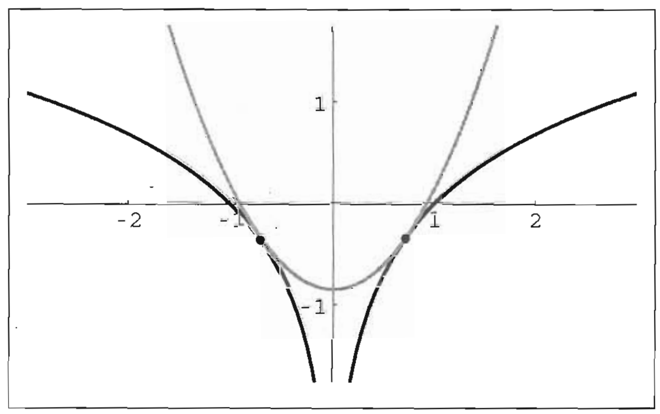
7. In the figure to the right the black curves comprise the graph of the function

$$y = \frac{1}{2}\ln(x^2).$$

The gray curve is the graph of the function

$$y = x^2 + c.$$

These graphs touch at the black points. Find the constant c and the coördinates of the black points.



8. A ball is launched vertically into the air and its distance from the ground (in feet) at any time $t \geq 0$ (t is in seconds) is given by $h(t) = \dfrac{100\,t}{1+t}$. The ball moves within a vertical pipe which is 100 feet long and which has a gate at the top end. The ball is equipped with a remotely operated cruise control device. That is we can fix the velocity of the ball at any moment.

   (a) The gate will open 10 seconds after the ball has been launched and stay open for a blink of an eye. At which earlier time $t = t_0$ should you fix the velocity of the ball in order for the ball to leave the pipe?

   (b) The gate will open τ (here $\tau > 0$) seconds after the ball has been launched and stay open for a blink of an eye.

      i. Find the formula for the corresponding earlier time $F(\tau)$ such that the ball whose velocity is fixed at the time $F(\tau)$ will leave the pipe at the time τ.

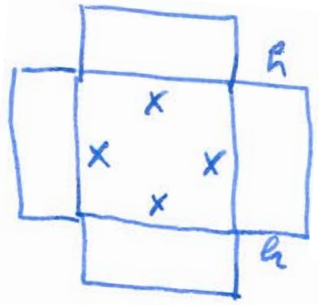
      ii. Taking the setting of this problem into account, answer the following question: Is the function $F(\tau)$ defined for all τ?

Illustrate with a picture.

1 a

$\boxed{1}$

$$\text{Area} = x^2 + 4x \cdot h = 1$$



So $h = \dfrac{1-x^2}{4x}$

$$\text{Volume} = x^2 \cdot h = x^2 \cdot \frac{1-x^2}{4x} = \frac{1}{4}x(1-x^2).$$

$V(x) = \frac{1}{4}x(1-x^2)$. ← a function of x

Find critical points. Note $0<x<1$.

$$V'(x) = \frac{1}{4}(1-x^2) + \frac{1}{4}x(-2x) = \frac{1}{4}(1-3x^2)$$

So $V'(x)=0$, for $\boxed{x = \frac{1}{\sqrt{3}}}$.

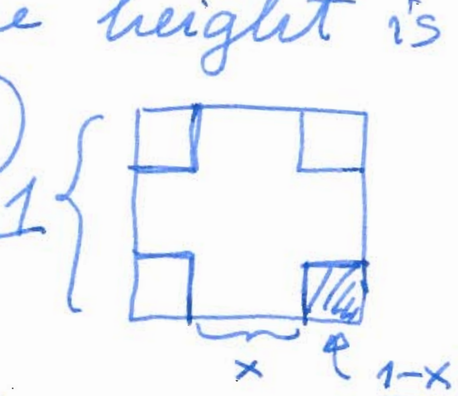
The $\boxed{\text{maximum volume}}$ is

$$V\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{4}\frac{1}{\sqrt{3}}\left(1-\frac{1}{3}\right) = \boxed{\frac{1}{6\sqrt{3}}}$$

the dimensions $\boxed{x = \frac{1}{\sqrt{3}}}$, $h = \dfrac{1-\frac{1}{3}}{4/\sqrt{3}} = \boxed{\frac{1}{2\sqrt{3}}}$

The height is one-half of the base side.

b



$$V(x) = x^2\,\frac{1-x}{2}$$

$$V'(x) = \cancel{2}x \cdot \frac{1-x}{\cancel{2}} - \frac{1}{2}x^2$$

$$= x - x^2 - \frac{1}{2}x^2$$

$$= x - \frac{3}{2}x^2 = x\left(1-\frac{3}{2}x\right)$$

So Vol. is max when $\boxed{x = 2/3}$

(b) The volume is maximum when $x = 2/3$ $h = 1/6$ $\boxed{2}$

$$V_{max} = \frac{4}{9}\cdot\frac{1}{6} = \boxed{\frac{2}{27}}$$

h is one quarter of the side of base

(c) In (a) volume $= \frac{1}{6\sqrt{3}}$

in (b) volume $= \frac{2}{27}$

The volume in (a) is "much" bigger

$\frac{1}{6\sqrt{3}} > \frac{2}{27}$, that is $27 > 12\sqrt{3}$

$9 > 4\sqrt{3}$

$81 > 16\cdot 3 = 48$

This makes sense since we are using all the material to make the box.

(2) $f(x) = \arctan(x) + \arctan\left(\frac{1}{x}\right)$

(a)

(b) The derivative should be 0.

(a)

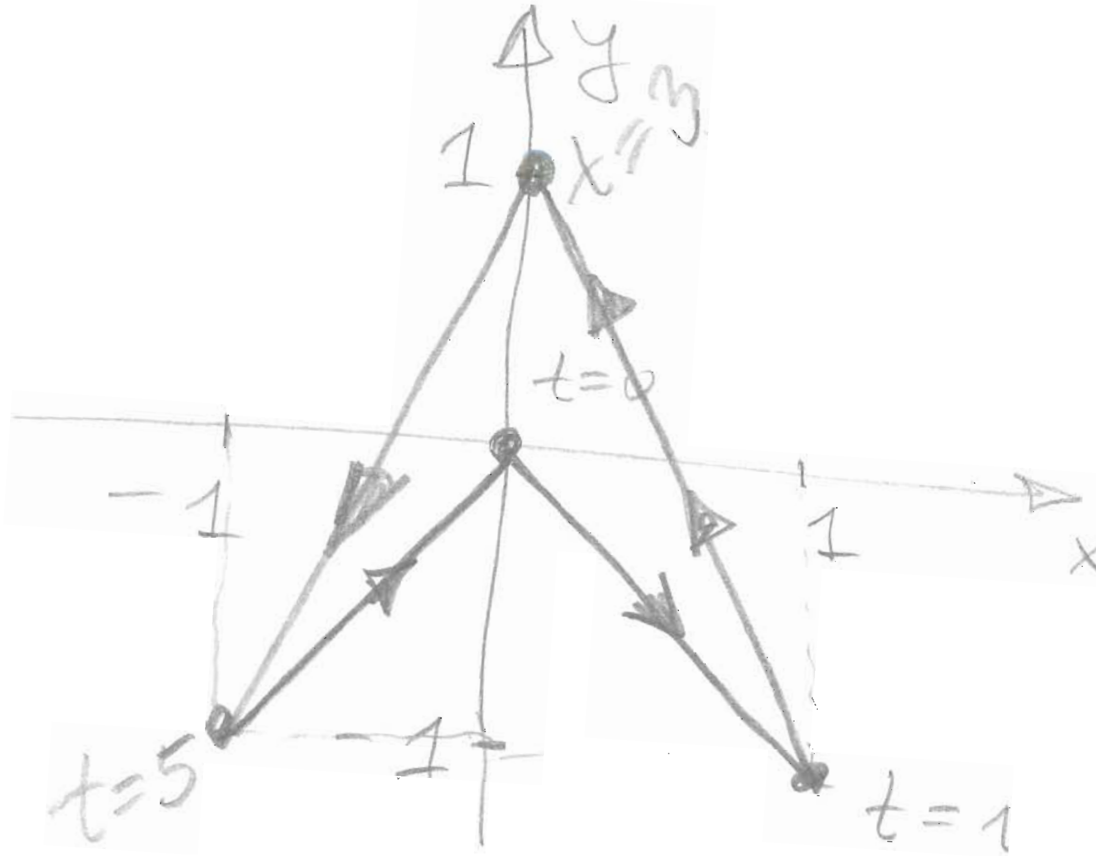
$$f'(x) = \frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right) + \frac{1}{1+x^2} = -\frac{\frac{1}{x^2}}{\frac{1+x^2}{x^2}} + \frac{1}{1+x^2}$$

ok

ok ✓ → $= 0$.

③

$\boxed{3}$



④ ⓐ $\dfrac{p(2)-p(0)}{2-0} = \dfrac{3\cdot 4-0}{2} = 6$

average velocity $\boxed{6\,mph}$

ⓑ $p'(t) = 6te^{2-t} + 3t^2e^{2-t}(-1)$

$= 3te^{2-t}(2-t) = 3te^{2-t}(2-t)$

$p''(t) = \dfrac{d}{dt}\left(3e^{2-t}(2t-t^2)\right) =$

$= 3e^{2-t}(-1)(2t-t^2) + 3e^{2-t}(2-2t)$

$= 3e^{2-t}\left(-2t+t^2+2-2t\right) =$

$= 3e^{2-t}(t^2-4t+2)$

4 c Maximize $\frac{p(t)}{t}$: $\boxed{3a}$

$$\frac{d}{dt}\left(\frac{p(t)}{t}\right) = \frac{d}{dt}\left(3te^{2-t}\right)$$

$$= 3e^{2-t} + 3te^{2-t}(-1)$$

$$= 3e^{2-t}(1-t)$$

So $t = 1$ is the only critical point.
It is a maximum

$$\frac{p(1)}{1} = 3\cdot 1\cdot e^{2-1} = 3e \approx 8.15485$$

4 b $p''(t) = 0$, solve $\boxed{4}$

$t^2 - 4t + 2 = 0$

$t_{1,2} = \dfrac{4 \pm \sqrt{16-8}}{2} = \dfrac{4 \pm 2\sqrt{2}}{2}$

$t_{1,2} = 2 \pm \sqrt{2}$ but $0 \le t \le 2$

So $\boxed{t = 2 - \sqrt{2}}$

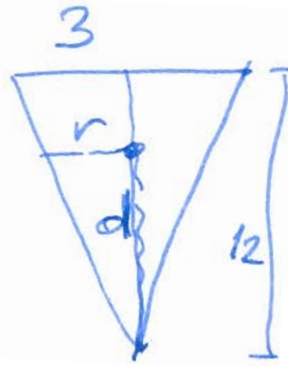
max velocity

$p'(2-\sqrt{2}) = 3(2-\sqrt{2})e^{\cancel{2}-\cancel{2}+\sqrt{2}}(\cancel{2}-\cancel{2}+\sqrt{2})$

max velocity $\boxed{3\sqrt{2}(2-\sqrt{2})e^{\sqrt{2}}}$

≈ 10.2226 mph

5 $\boxed{V(t) = (r(t))^2 \cdot d(t)\,\dfrac{\pi}{3}}$



$V'(t) = 4\ m^3/\text{hour}$

$\dfrac{d}{12} = \dfrac{r}{3}$

$\boxed{d = 4r}$

5 a

$r(t) = \frac{1}{4} d(t)$

$\boxed{5}$

$V(t) = \left(\frac{1}{4} d(t)\right)^2 d(t) \frac{\pi}{3}$

$V(t) = \frac{1}{16} \frac{\pi}{3} d(t)^3$

$V'(t) = \frac{\pi}{48} 3 d(t)^2 \cdot d'(t)$

$\| $

$4 = \frac{\pi}{48} * 3 * 8^2 \cdot \boxed{d'(t)}$

solve

$d'(t) = \frac{4 * 48}{\pi * 3 * 8^2} = \frac{\cancel{4 * 4 * 4 * 3}}{\pi * \cancel{3} * \cancel{4}^2 * \cancel{4}}$

$\boxed{d'(t) = 1/\pi}$

b

$d(t) = 4 r(t)$

$V(t) = (r(t))^2 \cdot 4 (r(t)) \frac{\pi}{3}$

$V(t) = \frac{4\pi}{3} r(t)^3$

$V'(t) = \frac{4\pi}{3} * 3 r(t)^2 * r'(t)$

5 b

$\boxed{6}$

$$4 = \frac{4\pi}{3} * 3\,(r(t))^2 * r'(t)$$

depth $= 8$ so $8 = 4r$

$$\boxed{r = 2}$$

so

$$\cancel{4} = \frac{\cancel{4}\pi}{\cancel{3}} * \cancel{3}\,(2)^2 * r'$$

$$\boxed{r'(t) = \frac{1}{4\pi}}$$

But the surface area is

$$A(t) = r(t)^2\,\pi$$

$$A'(t) = 2\underbrace{(r(t))}_{=2} \cdot \underbrace{r'(t)}_{} \,\pi$$

$$2 * 2 * \frac{1}{4\pi} * \pi$$

$$\boxed{A'(t) = 1 \ m^2/\text{hour}}$$

6 a $$f'(x) = \frac{1 * \sqrt{1+2x^2} - (1+x)\cancel{\frac{1}{2}} \frac{\overset{2}{\cancel{4}}x}{\sqrt{1+2x^2}}}{1+2x^2} \quad \boxed{7}$$

$$f'(x) = \frac{1+2x^2 - (1+x)\,2x}{(1+2x^2)^{3/2}}$$

$$f'(x) = \frac{1+\cancel{2x^2} - 2x - \cancel{2x^2}}{(1+2x^2)^{3/2}}$$

$$f'(x) = \frac{1-2x}{(1+2x^2)^{3/2}}$$

C.P. $\boxed{x = 1/2}$

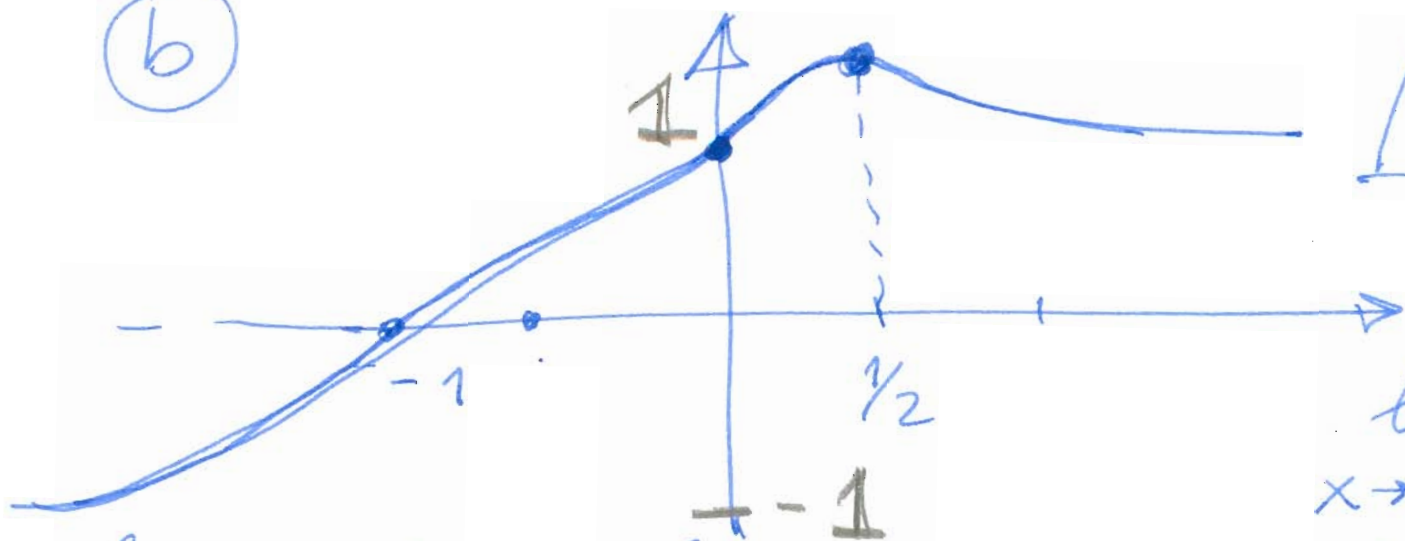
$x < 1/2$ $\qquad$ $x > 1/2$

$f' > 0$ $\qquad$ $f' < 0$

increasing $\qquad$ decreasing

$f(1/2)$ local max

(b)



$\boxed{8}$

$\lim_{x \to +\infty} = 1$

$\lim_{x \to -\infty} = -1$

based on graph

no global min

$$f(1/2) = \frac{3/2}{\sqrt{1+8\cdot 1/4}} = \boxed{\sqrt{\frac{3}{2}}}$$ global max

(c) $$f''(x) = \frac{-2(1+2x^2)^{3/2} - (1-2x)\frac{3}{2}(1+2x^2)^{1/2}\cdot 4x}{(1+2x^2)^3}$$

$$f''(x) = \frac{(1+2x^2)^{1/2}(-2-4x^2-6x+12x^2)}{(1+2x^2)^3}$$

$$f''(x) = \frac{+8x^2-6x-2}{(1+2x^2)^{5/2}}$$

Solve

9

$$8x^2-6x-2=0$$

$$x_{1,2} = \frac{+6 \pm \sqrt{36+4\cdot 2\cdot 8}}{16}$$

$$= \frac{6 \pm \sqrt{36+64}}{16}$$

$$= \frac{6 \pm 10}{16}$$

$$\boxed{x_1 = -\frac{1}{4}} \qquad \boxed{x_2 = 1}$$

concave up	concave dow
$\left(-\infty, -\frac{1}{4}\right)$	$\left(-\frac{1}{4}, 1\right)$
$(1, +\infty)$	

7

$$\frac{d}{dx}\left(\frac{1}{2}\ln x^2\right) =$$

$\boxed{10}$

$$= \frac{1}{\cancel{2}}\frac{1}{x^{\cancel{2}}}\cdot \cancel{2}\cancel{x} = \frac{1}{x}$$

$$\frac{d}{dx}(x^2+c) = 2x$$

find where $\frac{1}{x} = 2x$

$$2x^2 = 1$$

$$\boxed{x_{1,2} = \pm\frac{1}{\sqrt{2}}}$$

Touch at $x = \frac{1}{\sqrt{2}}$

$$\frac{1}{2}\ln\frac{1}{2} = \frac{1}{2} + c$$

So $c = \frac{1}{2}\left(\ln\frac{1}{2} - 1\right) = -\frac{1}{2}(1+\ln 2)$

coordinates

$$\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\ln\frac{1}{2}\right)$$

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\ln\frac{1}{2}\right)$$

11

8 a $\underline{h'(t)} = \frac{100}{(1+t)^2}$

$$h'(t_0) = \frac{100-h(t_0)}{10-t_0}$$

$$\frac{\cancel{100}^{1}}{(1+t_0)^{\cancel{2}}} = \frac{100 - \frac{100t_0}{1+t_0}}{10-t_0} = \frac{\frac{\cancel{100}^{1}}{\cancel{1+t_0}}}{10-t_0}$$

$$10-t_0 = 1+t_0$$

$$2t_0 = 9 \qquad \boxed{t_0 = 9/2}$$

b $h'(t) = \frac{100-h(t)}{\tau - t}$

$$\frac{\cancel{100}^{1}}{(1+t)^2} = \frac{100 - \frac{100t}{1+t}}{\tau - t} = \frac{\frac{\cancel{100}^{1}}{1+t}}{\tau - t}$$

$$\tau - t = 1 + t$$

$$2t = \tau - 1$$

$$t = \frac{\tau - 1}{2}$$

$$\boxed{F(\tau) = \frac{\tau - 1}{2}}$$

Since we need $F(\tau) \geq 0$

then $\tau \geq 1$