MATH 204 Final Examination March 17, 2011

Name Key

GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS.

1. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} -1 \\ 1 \\ 4 \\ 4 \end{bmatrix}$ be given vectors in $\mathbb{R}^4$.

   (a) Row reduce the 4×5 matrix whose columns are the given vectors. Use the reduced row echelon form to answer the questions below.

   (b) Are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ linearly independent? Give justification for your answer based on the definition. Be specific!

   (c) Find a basis for span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$. What is the dimension of this space? Do vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ span $\mathbb{R}^4$. Again, be specific!

   (d) What is the dimension of $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? What is the dimension of $W = \text{span}\{\mathbf{v}_4, \mathbf{v}_5\}$? Find a vector $\mathbf{u}$ which belongs to both subspace V and W. Be specific! Give $\mathbf{u}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and of $\mathbf{v}_4, \mathbf{v}_5$. A complete answer to (1b) can help here.

2. (a) Show that the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, are linearly independent.

   (b) Represent each of the vectors $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$.

   (c) It is given that $A\mathbf{u}_1 = \mathbf{u}_2$, $A\mathbf{u}_2 = \mathbf{u}_1$ and $A\mathbf{u}_3 = \mathbf{u}_3$. Find the matrix A. (Hint: (2b) is essential here.)

3. Consider the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$.

   (a) Use the method of row-reduction to find the inverse matrix of A.

   (b) Use your answer to (3a) to solve the matrix vector equation $x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

   (c) Suppose that B is a 3×3 matrix such that the matrix equation $AB\mathbf{v} = \mathbf{0}$ has more than one solution. What, if anything can you conclude about the invertibility of B? Justify your answer.

4. Consider the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 2 & 5 \end{bmatrix}$.

   (a) Find all eigenvalues of A.

   (b) For each eigenvalue find a basis for the corresponding eigenspace.

OVER

$\boxed{1}$

1. a)

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 & -1 \\ 1 & 1 & 0 & 1 & 1 \\ 2 & 0 & 2 & 1 & 4 \\ 2 & -1 & 3 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & 2 & 2 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & -1 & 1 & 2 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b) The vectors are linearly <u>dependent</u>!
To prove that we need a nontrivial linear combination which $= \vec{0}$.
For that we need to find Nul A.

$$\begin{aligned} x_1 \quad + x_3 \quad + x_5 &= 0 \\ x_2 - x_3 \quad - 2x_5 &= 0 \\ x_4 + 2x_5 &= 0 \end{aligned}$$

x_5 free
x_3 free

$x_1 = -x_3 - x_5$
$x_2 = x_3 + 2x_5$
$x_3 \to$ free
$x_4 = -2x_5$
$x_5 \to$ free

$$\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -1 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} x_5$$

1 b Thus

$$-\vec{v}_1 + 2\vec{v}_2 - 2\vec{v}_4 + \vec{v}_5 = \vec{0}$$

$$-\underset{\vec{v}_1}{\begin{bmatrix}1\\1\\2\\2\end{bmatrix}} + 2\underset{\vec{v}_2}{\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}} - 2\underset{\vec{v}_4}{\begin{bmatrix}-1\\1\\1\\0\end{bmatrix}} + \underset{\vec{v}_5}{\begin{bmatrix}-1\\1\\4\\4\end{bmatrix}} = \vec{0}$$

TRUE!

1 c $\vec{v}_1, \vec{v}_2, \vec{v}_4$ span the given space

The RREF tells that they are linearly independent and

$\vec{v}_3 = \vec{v}_1 - \vec{v}_2$ and

$\vec{v}_5 = \vec{v}_1 - 2\vec{v}_2 + 2\vec{v}_4$

1 d $\dim V = 2$ $\dim W = 2$

$$\underbrace{-\vec{v}_1 + 2\vec{v}_2}_{\in V} = \underbrace{2\vec{v}_4 - \vec{v}_5}_{\in W} = \vec{u}$$

(2) (a) The following row reduction will be useful: $\boxed{3}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

(b) So $\vec{e}_1 = -\vec{u}_2 + \vec{u}_3$ True

$\vec{e}_2 = +\vec{u}_1 + \vec{u}_2 - \vec{u}_3$ True

$\vec{e}_3 = -\vec{u}_1 + \vec{u}_3$ True

(2) (a) $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are linearly independent since $x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3$ has only the trivial solution.

(2c) $A\vec{e}_1 = -A\vec{u}_2 + A\vec{u}_3 = -\vec{u}_1 + \vec{u}_3 = \vec{e}_3$

$A\vec{e}_2 = \vec{u}_2 + \vec{u}_1 - \vec{u}_3 = \vec{e}_2$

$A\vec{e}_3 = -\vec{u}_2 + \vec{u}_3 = e_1$

$\boxed{4}$

2 c Thus

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad !$$

3 a

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 3 & 2 & -2 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -2 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -3 & 1 \\ 0 & 1 & 0 & -2 & -2 & 1 \\ 0 & 0 & 1 & 2 & 3 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -1 & -3 & 1 \\ -2 & -2 & 1 \\ 2 & 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ok

3 b

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \\ 5 \end{bmatrix}$$

$x_1 = -4, \; x_2 = -3, \; x_3 = 5$

3 c $AB\vec{v} = \vec{0}$ $\boxed{5}$

Since A is invertible yields

$$B\vec{v} = \vec{0}$$

Since $\vec{v}$ is non zero vector the columns of B are linearly dependent thus B is NOT invertible.

4 a $$\begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 4-\lambda & 1 \\ 0 & 2 & 5-\lambda \end{vmatrix} = (3-\lambda)\big((4-\lambda)(5-\lambda)-2\big)$$

$$= (3-\lambda)(\lambda^2 - 9\lambda + 20 - 2)$$

$$= (3-\lambda)(\lambda^2 - 9\lambda + 18)$$

$$= (3-\lambda)(\lambda-3)(\lambda-6)$$

Thus $\lambda = 3$ and $\lambda = 6$ are eigenvalues.

4 (b) Solve $(A-3I)\vec{x}=0$ $\boxed{6}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_1 free
x_3 free
$x_2 = -x_3$

$$\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

thus the basis for the eigenspace is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

Solve $(A-6I)\vec{x}=0$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$
$x_2 = 1/2\, x_3$
x_3 free

$$\vec{x} = x_3 \begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix}$$

~~The~~ a basis is $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

5 a $\dim \mathbb{P}_2 = 3$ $\boxed{7}$

a basis is

$p_1(t) = 1, \; p_2(t) = t, \; p_3(t) = t^3$

these vectors span $\mathbb{P}_2$ by ~~the~~ its definition

$at^2 + bt + c = a p_3(t) + b p_2(t) + c p_1(t)$

these polynomials are linearly independent

since

$x_1 p_1(t) + x_2 p_2(t) + x_3 p_3(t) = 0$ for all t

yields $x_1 + x_2 t + x_3 t^2 = 0$ for all t

Set $t = 0$; $t = -1$ and $t = 1$

we get
$$x_1 = 0$$
$$-x_2 + x_3 = 0$$
$$x_2 + x_3 = 0$$

thus
$$x_1 = 0$$
$$2x_3 = 0$$
$$2x_2 = 0$$

So $x_1 = x_2 = x_3 = 0$

5 b Set

$\boxed{8}$

$$x_1 q_1(t) + x_2 q_2(t) + x_3 q_3(t) = 0$$

for all t

For $t=0$, $t=+1$, $t=2$ this yields

$$x_2 + x_3 = 0$$

$$+x_1 = 0$$

$$2x_1 - 3x_2 + 5x_3 = 0$$

Thus $x_1 = 0$ and $x_2 + x_3 = 0$

$-3x_2 + 5x_3 = 0$

Solving for x_2, x_3 yields $x_2 = x_3 = 0$.

This prove linear independence.

5 c H is a subspace since

$u, v \in H$ yields $u(-1) = u(1)$

and $u(-1) = v(1)$

Now calculate

$(u+v)(-1) = u(-1) + v(-1) = u(1) + v(1)$

$= (u+v)(1)$

So $u + v \in H$. Similarly for

cu for any $c \in \mathbb{R}$.

5 a) By b) q_2 and q_3 are linearly independent and $q_2(-1)=q_2(1)=0$ and $q_3(-1)=q_3(1)=0$. $\boxed{9}$

Thus $q_2, q_3 \in H$. But $q_1 \notin H$

Therefore H is a proper part of $\mathbb{P}_2$.

The basis of H is $\{q_2, q_3\}$ and $\dim H = 2$.

6 a) S cannot be 1-to-1. We know that $\text{rank}\, A + \dim \text{Nul}\, A = 17$ and $\text{rank}\, A \leq 12$. Thus

$$\dim \text{Nul}\, A \geq 5.$$

Thus S cannot be one-to-one (meaning $\text{Nul}\, A = \{0\}$)

Yes S can be onto. In this case $\text{rank}\, A = 12$ and

$$\dim \text{Nul}\, A = 5$$

⑥ ⓑ T <u>can be 1-to-1.</u> $\boxed{10}$

In this case $\mathrm{Nul}\, B = \{\vec{0}\}$

T <u>cannot be onto</u> since

$\mathrm{rank}\, B + \dim \mathrm{Nul}\, B = 12$

So $\mathrm{rank}\, B \leq 12$ and

$\dim \mathbb{R}^{17} = 17$

T is 1-to-1 if and only if

$\mathrm{Nul}\, B = \{\vec{0}\}$.

⑥ ⓒ ~~This means that T is not onto.~~

~~that~~ This means that T is not onto.

Thus $\mathrm{rank}\, A < 12$

Since $\mathrm{rank}\, A + \dim \mathrm{Nul}\, A = 17$

we conclude that

$6 \leq \dim \mathrm{Nul}\, A \leq 17$

6 d ~~Similarly to e~~

~~rank B~~

We cannot conclude anything from this information, since it is always true that there is $\vec{b} \in R^{17}$ such that $B\vec{x} = \vec{b}$ has no solution.

~~Ea~~ T is <u>never</u> onto.

$$0 \le \dim \operatorname{Nul} B \le 12$$

6 e $\operatorname{Nul} A \neq \{\vec{0}\}$.

Both matrices $\underbrace{AB}_{12 \times 12}$ and $\underbrace{BA}_{17 \times 17}$ are defined. But

$$\operatorname{Nul}(BA) \neq \{\vec{0}\}$$

Thus BA cannot be invertible.

AB can be invertible.

$\boxed{12}$

7 a

$$\begin{bmatrix} 4/5 & 2/5 \\ 1/5 & 3/5 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = 1 \cdot \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

$$8/15 + 2/15 = \frac{10}{15} = \frac{2}{3} = \quad \frac{2}{3} \cdot \frac{6}{}$$

$$\frac{2}{15} + \frac{3}{15} = \quad \frac{5}{15}$$

$$\cancel{\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ \lambda \end{bmatrix}}$$

Yes, the correspondin e.value is $\underline{1}$

b

$$\begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - 2$$

$$= \lambda^2 - 7\lambda + 12 - 2$$

$$= \lambda^2 - 7\lambda + 10$$

the other eigenvalue is $2/5$

$$\begin{bmatrix} 4-2 & 2 \\ 1 & 3-2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

the other eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

7 c $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = c_1 \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\boxed{13}$

$$\left[\begin{array}{cc|c} 2/3 & 1 & 1/2 \\ 1/3 & -1 & 1/2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & 3/2 \\ 2 & 3 & 3/2 \end{array}\right]$$

$\cancel{-6/2}$

$-6/2 + 3/2$

$$\sim \left[\begin{array}{cc|c} 1 & -3 & 3/2 \\ 0 & +9 & -3/2 \end{array}\right]$$

$$9c_2 = -3/2$$
$$c_2 = -1/6$$

$$c_1 + 3\,1/6 = 3/2$$
$$c_1 = 3/2 - 1/2 = 1$$

$$\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

7 d $$A x_0 = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} - \frac{1}{6}\frac{2}{5}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$ $\boxed{14}$

$$= \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} - \frac{1}{15}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 9/15 \\ 6/15 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}$$

$$A^2 x_0 = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} - \frac{1}{6}\left(\frac{2}{5}\right)^2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} - \frac{2}{3 \cdot 25}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 48/75 \\ 27/75 \end{bmatrix} \quad \ldots$$

7 e $$A^k \vec{x}_0 = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} - \frac{1}{6}\left(\frac{2}{5}\right)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for large k $A^k \vec{x}_0$ is very close to $\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$.