MATH 204 Examination November 23, 2019 Name ______________________

Give detailed explanations for your answers.

On in class exams I assign four problems. Each is worth 25 points.
I try to assign problems from different topics that we covered.
Below are several problems to help you get used to my style of exam questions. Some of these problems are too long to be on an exam.

1. Consider the matrix $V = \begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix}$

   (a) Calculate the determinant $\det V$.

   (b) Find values for x, y, z such that V is invertible and values for x, y, z in which V is not invertible.

2. Calculate the determinant $\begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 5 & 0 & 6 & 7 \\ 0 & 0 & 8 & 9 \end{vmatrix}$.

3. Determine h such that

$$\begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 1 & h \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = 0.$$

4. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_5 = \begin{bmatrix} -1 \\ 1 \\ 4 \\ 4 \end{bmatrix}$ be given vectors in $\mathbb{R}^4$.

   (a) Row reduce the 4×5 matrix whose columns are the given vectors. Use the reduced row echelon form to answer the questions below.

   (b) Are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ linearly independent? Give justification for your answer based on the definition. Be specific!

   (c) Find a basis for $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$. What is the dimension of this space? Do vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ span $\mathbb{R}^4$. Again, be specific!

   (d) What is the dimension of $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? What is the dimension of $W = \operatorname{span}\{\mathbf{v}_4, \mathbf{v}_5\}$? Find a <u>nonzero</u> vector $\mathbf{u}$ which belongs to both subspace V and W. Be specific! Give $\mathbf{u}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and of $\mathbf{v}_4, \mathbf{v}_5$. A complete answer to (4b) can help here.

5. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 5 \\ 1 & 0 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & -1 \\ -2 & -1 & 0 \\ -7 & 4 & 3 \end{bmatrix}$. Determine whether the matrices A and B have the same column space. (*Important Note:* If you claim that A and B have the same column space justify your claim by calculations. If you claim that the matrices do not have the same column space give a specific vector which is in one column space but not in the other.)

6. (a) Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be vectors in a vector space $\mathcal{V}$. State the definition of linear independence for the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$

   The rest of this problem is about the space $\mathbb{P}_2$ of all polynomials of degree at most 2.

   (b) Consider the polynomials $\mathbf{q}_1(t) = t, \mathbf{q}_2(t) = 1 - t^2, \mathbf{q}_3(t) = 1 + t^2$ in $\mathbb{P}_2$. Are these polynomials linearly independent in $\mathbb{P}_2$? Why or why not?

   (c) Let $\mathcal{H}$ be the set of all polynomials $\mathbf{p}$ in $\mathbb{P}_2$ such that $\mathbf{p}(-1) = \mathbf{p}(1)$. Show that $\mathcal{H}$ is a subspace of $\mathbb{P}_2$.

   (d) Find a basis for $\mathcal{H}$.

7. Suppose that A is 12×17 matrix and let $S : \mathbb{R}^{17} \to \mathbb{R}^{12}$ be a linear transformation given by $S(\mathbf{x}) = A\mathbf{x}$.

   Suppose that B is 17×12 matrix and let $T : \mathbb{R}^{12} \to \mathbb{R}^{17}$ be a linear transformation given by $T(\mathbf{x}) = B\mathbf{x}$.

   (a) Can S be one-to-one? Why or why not? Can S be onto? Why or why not? Is there a connection between S being one-to-one or onto and $\operatorname{Nul} A$?

   (b) Can T be one-to-one? Why or why not? Can T be onto? Why or why not? Is there a connection between T being one-to-one or onto and $\operatorname{Nul} B$?

   (c) If $\mathbf{b} \in \mathbb{R}^{12}$ is such that $A\mathbf{x} = \mathbf{b}$ has no solution, what can you conclude about the dimension of $\operatorname{Nul} A$? (Give your answer in the form: $?? \leq \dim \operatorname{Nul} A \leq ??$, where ?? stand for a specific integer.)

   (d) If $\mathbf{b} \in \mathbb{R}^{17}$ is such that $B\mathbf{x} = \mathbf{b}$ has no solution, what can you conclude about the dimension of $\operatorname{Nul} B$? (Give your answer in the form: $?? \leq \dim \operatorname{Nul} B \leq ??$, where ?? stand for a specific integer.)

   (e) Which of the matrices AB, BA is defined? Which of these matrices could be invertible? Which of these matrices is never invertible? Explain your answers.

8. In this problem we consider the vector space $\mathbb{P}_2$ of all polynomials of degree at most 2. Recall that the standard basis for $\mathbb{P}_2$ is $\mathcal{S} = \{1, t, t^2\}$.

   (a) Consider the polynomials $\mathbf{q}_1(t) = 1 - t^2, \mathbf{q}_2(t) = t, \mathbf{q}_3(t) = 1 + t^2$ in $\mathbb{P}_2$. Prove that $\mathcal{B} = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ is a basis for $\mathbb{P}_2$.

   (b) Find $\underset{\mathcal{B}\leftarrow\mathcal{S}}{P}$.

   (c) Find the basis $\mathcal{C}$ for $\mathbb{P}_2$ if $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = A$, where $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$.

9. Below I give two 3×5 matrices A and B together with their reduced row echelon forms:

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 1 & 2 & 2 & 4 & 5 \\ 2 & 4 & 1 & 5 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 3 & 1 & 2 & 1 \\ 5 & 4 & 2 & 2 & 1 \\ 7 & 5 & 1 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 2 & 1 \\ 0 & 1 & 3 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Denote the columns of A by $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$, $\vec{v}_5$. Notice that the columns of the matrix B are the columns of the matrix A in reversed order. Therefore $\operatorname{Col} A = \operatorname{Col} B$.

(a) i. Which basis for $\operatorname{Col} A$ is determined by the given RREF of A? Call this basis $\mathcal{A}$.
    ii. Which basis for $\operatorname{Col} B$ is determined by the given RREF of B? Call this basis $\mathcal{B}$.

(b) With the bases $\mathcal{A}$ and $\mathcal{B}$ found in the previous two items, find $\underset{\mathcal{A}\leftarrow\mathcal{B}}{P}$ and $\underset{\mathcal{B}\leftarrow\mathcal{A}}{P}$.

10. By $\mathbb{R}^{2\times 2}$ we denote the vector space of all 2×2 matrices. Let A be a matrix in $\mathbb{R}^{2\times 2}$. Decide which of the following are subspaces and justify your answer.

    (a) $\mathcal{H} = \{B \in \mathbb{R}^{2\times 2} : AB = BA\}$
    (b) $\mathcal{H} = \{B \in \mathbb{R}^{2\times 2} : BA = 0\}$
    (c) $\mathcal{H} = \{B \in \mathbb{R}^{2\times 2} : BB^\top = 0\}$
    (d) $\mathcal{H} = \{B \in \mathbb{R}^{2\times 2} : \det(B) = 0\}$

11. Consider the following matrices

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 3 & 2 & 3 \\ 4 & 3 & 6 \end{bmatrix}$$

Determine all vectors which belong to both $\operatorname{Col} A$ and $\operatorname{Col} B$.

12. Consider the vectors: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

    (a) Calculate the dimension of $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.
    (b) Let A be a matrix such that $\operatorname{Nul} A = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Based on what you calculated in (12a), can you determine what is $\operatorname{rank} A$? Explain your reasoning.

13. Consider the vectors: $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$. These three vectors form a basis for $\mathbb{R}^3$. Denote this bases by $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

    (a) Which vector $\mathbf{v} \in \mathbb{R}^3$ satisfies $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$?

    (b) Calculate $[\mathbf{e}_1]_{\mathcal{B}}$, $[\mathbf{e}_2]_{\mathcal{B}}$, $[\mathbf{e}_3]_{\mathcal{B}}$, where $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

14. (a) Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be vectors in a vector space $\mathcal{V}$. State the definition of linear independence for the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$

The rest of this problem is about the space $\mathbb{P}_2$ of all polynomials of degree at most 2.

(b) Consider the polynomials $\mathbf{q}_1(t) = t, \mathbf{q}_2(t) = 1 - t^2, \mathbf{q}_3(t) = 1 + t^2$ in $\mathbb{P}_2$. Are these polynomials linearly independent in $\mathbb{P}_2$? Why or why not?

(c) Let $\mathcal{H}$ be the set of all polynomials $\mathbf{p}$ in $\mathbb{P}_2$ such that $\mathbf{p}(-1) = \mathbf{p}(1)$. Show that H is a subspace of $\mathbb{P}_2$.

(d) Find a basis for $\mathcal{H}$.

15. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 5 \\ 1 & 0 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & -1 \\ -2 & -1 & 0 \\ -7 & 4 & 3 \end{bmatrix}$. Determine whether the matrices A and B have the same column space. (*Important Note:* If you claim that A and B have the same column space justify your claim by calculations. If you claim that the matrices do not have the same column space give a specific vector which is in one column space but not in the other.)

16. Let $\mathbb{P}_3$ be the vector space of polynomials of degree less or equal to three. Recall that the polynomials 1, t, t^2, t^3 form a basis for $\mathbb{P}_3$, which we denote by $\mathcal{S}$ and call the standard basis for $\mathbb{P}_3$.

Let $\mathcal{H}$ be the subset of $\mathbb{P}_3$ consisting of those polynomials $\mathbf{p}(t)$ satisfying $\mathbf{p}(0) = 0$ and $\mathbf{p}(1) = 0$.

(a) Prove that $\mathcal{H}$ is a subspace of $\mathbb{P}_3$.

(b) Let $\mathbf{u}(t) = t^3 - t^2$ and $\mathbf{v}(t) = t^3 - t$ be two polynomials in $\mathbb{P}_3$. Let $\mathcal{B} = \{\mathbf{u}, \mathbf{v}\}$.

State the two properties that $\mathcal{B}$ must satisfy in order to be a basis for $\mathcal{H}$ and prove them.