MATH 204 Examination January 23, 2022

Name ______________________________

GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS.
On in-class exams I assign four problems. Each is worth 25 points.
I try to assign problems from different topics that we covered.
Below are several problems to help you get used to my style of exam questions.

1. Consider the following nonhomogeneous linear system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 3 \\ -x_1 \qquad\quad + x_3 + 2x_4 &= -1 \\ 2x_1 + x_2 \qquad\quad - x_4 &= 3 \end{aligned}$$

   (a) Write the augmented matrix of this system and row reduce it to the reduced row echelon form.

   (b) Write the solution of the given system in parametric vector form.

   (c) Without doing any additional row reduction write the solution of the corresponding homogeneous equation in parametric vector form.

2. Consider the system of equations

$$\begin{aligned} ax + y &= a^2 \\ x + ay &= 1 \end{aligned}$$

   (a) Find the value of a for which the above system is inconsistent.

   (b) Find the value of a for which the above system has infinitely many solutions.

   (c) Find all values of a for which the above system has unique solution.

3. Consider the chemical equation

$$\mathrm{Al_2O_3 + C \to Al + CO_2}$$

   which states that aluminum oxide and carbon react to create elemental aluminum and carbon dioxide.

   (a) Find a system of linear equations whose solutions balance the above chemical equation.

   (b) Find the smallest integers which balance the above chemical equation.

4. Let A be a 4×3 matrix and let $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

   (a) Suppose that the matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution. What can you say about linear dependence/independence of the columns of A? Justify your answer.

   (b) Suppose that the matrix equation $A\mathbf{x} = \mathbf{b}$ has at least two solutions. What can you say about linear dependence/independence of the columns of A? Justify your answer.

5. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 3 \\ 2 & 1 & 3 \end{bmatrix}$.

   (a) Can the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ be written as a linear combination of the columns of A? If your answer is yes, then write $\mathbf{b}$ as a linear combination of the columns of A.

   (b) Do the columns of A span $\mathbb{R}^3$? If your answer is no, then provide one specific vector in $\mathbb{R}^3$ which cannot be written as a linear combination of the columns of A.

6. It is given that the matrix A is row equivalent to

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   (a) Write the general solution of the equation $A\mathbf{x} = \mathbf{0}$ in parametric vector form.

   (b) Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for every $\mathbf{b}$ in $\mathbb{R}^4$? Explain your answer.

7. (a) Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. Complete the sentence: The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent if ....

   (b) Consider the vectors $\mathbf{v}_1 = \begin{bmatrix} 8 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} h \\ 5 \\ 4 \end{bmatrix}$. Determine for what values of the parameter h in $\mathbf{v}_3$ the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent.

   (c) Determine if the columns of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ are linearly independent.

8. (a) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the x_2-axes and then reflects points through the line $x_2 = x_1$. Find the standard matrix of this linear transformation.

   (b) Show that T also can be described as a linear transforation that rotates points about the origin. What is the angle of that rotation?

9. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & c & 1 & 2 \\ 3 & 4 & 1 & 1 \end{bmatrix}.$$

   (a) For each c calculate the reduced row echelon form of A. (Hint: there are only two distinct cases.)

   (b) In each case above state whether the columns of A span $\mathbb{R}^3$.

10. I considered a matrix equation $A\mathbf{x} = \mathbf{0}$ with a 3×4 matrix A. I found that the vector parametric form of the solution of the given vector equation is

$$\mathbf{x} = s\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix},$$

where s and t are any numbers. Find the reduced row echelon form of the matrix A.

11. It is given that A is a 2×3 matrix and that

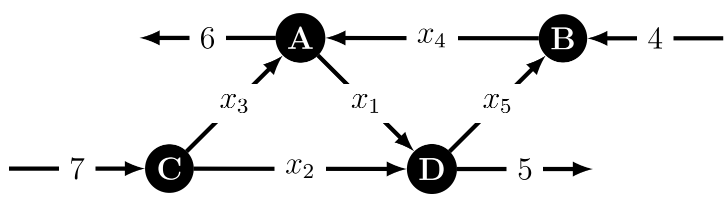
$$A\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Find all solutions of the matrix equation

$$A\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Write the solution set in parametric vector form.

12. Consider the network flow diagram below where the numbers indicate known flows, and x_1, x_2, x_3, x_4, x_5 indicate unknown flows on the corresponding edges.



(a) Write the linear system that models the above network flow.

(b) Find the Reduced Row Echelon Form of the augmented matrix of the system in the preceding item.

(c) Find the solution set and write it in the parametric vector form.

(d) Assume that all the flows are nonnegative integers. What restriction does this condition put on flows x_2 and x_3?

13. The matrix A and its reduced row echelon form are given below.

$$A = \begin{bmatrix} 1 & 1 & 3 & 2 & 1 & 1 \\ 1 & 2 & 5 & 3 & 0 & 3 \\ 2 & 3 & 8 & 5 & 1 & 4 \\ 2 & 2 & 6 & 4 & 3 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Denote the columns of A by $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$, $\mathbf{a}_5$, $\mathbf{a}_6$, that is

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \ \mathbf{a}_3 = \begin{bmatrix} 3 \\ 5 \\ 8 \\ 6 \end{bmatrix}, \ \mathbf{a}_4 = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 4 \end{bmatrix}, \ \mathbf{a}_5 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \ \mathbf{a}_6 = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 1 \end{bmatrix}.$$

(a) Based on the given RREF identify the maximum set of linearly independent vectors among $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$, $\mathbf{a}_5$, $\mathbf{a}_6$.

(b) Is $\mathbf{a}_2$ a linear combination of $\mathbf{a}_1$? Explain why.

(c) Is $\mathbf{a}_3$ a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$? Explain why.

(d) Is $\mathbf{a}_4$ a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$? Explain why.

(e) Is $\mathbf{a}_5$ a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{a}_4$? Explain why.

(f) Is $\mathbf{a}_6$ a linear combination of some of the preceding vectors? Which ones? Explain why.