MATH 209

Examination 1
January 25, 2011

Name Key

1. Let a be the compound proposition $(p \to q) \wedge (q \to r)$. Let b be the compound proposition $p \to r$.

   (a) Are a and b logically equivalent? If you claim that a and b are logically equivalent prove it by truth tables. If you claim that they are not logically equivalent give an example which shows that.

   (b) Is the implication $a \to b$ true? If true, prove it; if not state and prove its negation.

2. Let $p \to q$ be an implication. This is the original implication.

   (a) State clearly what is the **contrapositive**, what is the **converse** and what is the **inverse** of this implication. In each case give a formula for the implication. The formula should be an implication using two of the propositions $p, q, \neg p, \neg q$.

   (b) Based on the definitions in (2a) it should be clear that **the contrapositive of the contrapositive is the original implication**. State clearly (in English not using formulas) what is

      i. the contrapositive of the converse,
      ii. the contrapositive of the inverse,
      iii. the converse of the contrapositive,
      iv. the converse of the converse,
      v. the converse of the inverse,
      vi. the inverse of the contrapositive,
      vii. the inverse of the converse,
      viii. the inverse of the inverse.

   (c) Do you notice any patterns in (2b)? State clearly what you notice.

   (d) Based on the patterns discovered in (2c) decide (and explain how you decided) what is:
   the contrapositive of the converse of the inverse of the contrapositive of the inverse of the converse of the inverse of the contrapositive of the inverse of the contrapositive of the converse of the inverse of the converse of the contrapositive

3. Let $P(x, y)$ be a propositional function of two variables x and y. As usual $\neg P(x, y)$ denotes the negation of the proposition $P(x, y)$. By binding the variables x and y with quantifiers we get the following sixteen propositions:

a	$\forall x\, \forall y\; P(x,y)$	e	$\exists x\, \forall y\; P(x,y)$	i	$\forall y\, \forall x\; P(x,y)$	m	$\exists y\, \forall x\; P(x,y)$
b	$\forall x\, \forall y\; \neg P(x,y)$	f	$\exists x\, \forall y\; \neg P(x,y)$	j	$\forall y\, \forall x\; \neg P(x,y)$	n	$\exists y\, \forall x\; \neg P(x,y)$
c	$\forall x\, \exists y\; P(x,y)$	g	$\exists x\, \exists y\; P(x,y)$	k	$\forall y\, \exists x\; P(x,y)$	o	$\exists y\, \exists x\; P(x,y)$
d	$\forall x\, \exists y\; \neg P(x,y)$	h	$\exists x\, \exists y\; \neg P(x,y)$	l	$\forall y\, \exists x\; \neg P(x,y)$	p	$\exists y\, \exists x\; \neg P(x,y)$

   Some of these propositions are related. For example a and i are logically equivalent; also, h is the negation of a. That is $i \equiv a$ and $h \equiv \neg a$.

   (a) Identify six propositions in the above list which can be used to express the remaining ~~twelve~~ ten propositions as logically equivalent to one of the six propositions or their negations. Write all twelve relationships. (I already wrote two.)

   (b) Let $P(x, y)$ be the statement "$x < y$" and let the universe of discourse for the variables x and y be the set of all real numbers. Prove each of the six statements listed in (3a).

4. The universe of discourse in this problem consists of all integers. Prove the following two implications.

   (a) If a is odd and $b \neq 0$, then $\left(\frac{a}{b}\right)^2 \neq 2$.

   (b) If a is even and b is odd, then $\left(\frac{a}{b}\right)^2 \neq 2$.

$\boxed{1}$

(1) (a) a is $(p \to q) \wedge (q \to r)$

b is $p \to r$

These are not logically equivalent since for p T, q F, r T a is F and b is T.

(b) $a \to b$ is true. Here is a proof.

Assume a. Now I have to prove b. For that assume p. By a, or since a is true $p \to q$ and $q \to r$ are T. Since p and $p \to q$ are true q is true. Since q and $q \to r$ are true r is true. Thus $p \to q$ is proved.

(2) The contrapositive : $\neg q \to \neg p$

The converse $q \to p$

The inverse $\neg p \to \neg q$

(i) the inverse

(ii) the converse

(iii) the inverse

(iv) the original implication

(v) the contrapositive

(vi) the converse

(vii) the contrapositive

(viii) the original

(2c) The rule is: if two are different the result is the third one. If two are the same the result is the original implication. $\boxed{2}$

(2d) **Count:** the contrapositive 5
the converse 4
the inverse 5

So the result is the <u>converse</u>.

(3a)

* a	* k
* b	* l
* c	$m \equiv \neg l$
* d	$n \equiv \neg k$
$e \equiv \neg d$	$o \equiv \neg b$
$f \equiv \neg c$	$p \equiv \neg a$
$g \equiv \neg b$	
$h \equiv \neg a$	
* $i \equiv a$	
$j \equiv b$	

six

3

3b a $\forall x \forall y\,(x < y)$

not true. Prove the negation

$\exists x \exists y \quad x \geq y$ Yes take $x = 1,\ y = 0$

b $\forall x \forall y \quad x \geq y$ Not true

Prove the negation

$\exists x \exists y \quad x < y$. Take $x = 0,\ y = 1$

c $\forall x \exists y \quad x < y$ Yes take $y = x + 1$

d $\forall x \exists y \quad x \geq y$ Yes take $y = x - 1$

k $\forall y \exists x \quad x < y$ Yes take $x = y - 1$.

e $\forall y \exists x \quad x \geq y$ Yes, take $x = y + 1$.