MATH 209

Examination 3
March 4, 2011

Name ____________________

1. Use mathematical induction to prove the following statement

$$\sum_{k=1}^{n} \frac{1}{k^2} < 2 - \frac{1}{n} \qquad \text{for all} \qquad n \in \mathbb{Z}_+.$$

2. The universe of discourse in this problem is the set of positive integers. Consider the recursively defined sequence:

$$f(1) = 1, \quad f(n) = n - f(n-1), \quad n \geq 2$$

   (a) i. Write the first 10 elements of this sequence.
   ii. Guess the formula for the sequence as a closed form expression of n.

   (b) Prove the formula that you guessed.

3. In this problem we consider bit strings of length 7.

   (a) What is cardinality of the set of all bit strings of length seven?

   (b) How many bit strings of length seven have exactly three consecutive 0s?

   (c) How many bit strings of length seven have at least three consecutive 0s?

   (d) How many bit strings of length seven have at least three consecutive 0s and at least three consecutive 1s?

   (e) How many bit strings of length seven have either at least three consecutive 0s or at least three consecutive 1s?

4. The universe of discourse in this problem is the set of integers. In this problem we study devisability by 4. Recall that when divided by 4 an integer leaves a remainder $0, 1, 2$ or 3.

   (a) Given any five integers a, b, c, d, e prove that there exists two of them whose difference is divisible by 4.

   (b) Given any five integers a, b, c, d, e prove or disprove that there exists two of them whose sum is divisible by 4.

   (c) Given any four integers a, b, c, d prove that there exists two of them whose sum or difference is divisible by 4.

   This problem involves the Pigeonhole principle. Please be clear what are pigeons and what are pigeonholes in your solutions.