along the x-axis near $(0,0)$, then the values $g(x,0)$ are equal to 1, while if we consider points $(0,y)$ lying along the y-axis near $(0,0)$, then the values $g(0,y)$ are equal to 0. Thus, within any distance (no matter how small) from the origin, there are points where $g=0$ and points where $g=1$. Therefore the limit $\lim_{(x,y)\to(0,0)} g(x,y)$ does not exist, and thus g is not continuous at $(0,0)$.

While the notions of limit and continuity look formally the same for one- and two-variable functions, they are somewhat more subtle in the multivariable case. The reason for this is that on the line (1-space), we can approach a point from just two directions (left or right) but in 2-space there are an infinite number of ways to approach a given point.

Exercises and Problems for Section 12.6

Exercises

Are the functions in Exercises 1-6 continuous at all points in the given regions?

1. $\dfrac{1}{x^2+y^2}$ on the square $-1 \le x \le 1, -1 \le y \le 1$
2. $\dfrac{1}{x^2+y^2}$ on the square $1 \le x \le 2, 1 \le y \le 2$
3. $\dfrac{y}{x^2+2}$ on the disk $x^2+y^2 \le 1$
4. $\dfrac{e^{\sin x}}{\cos y}$ on the rectangle $-\frac{\pi}{2} \le x \le \frac{\pi}{2}, 0 \le y \le \frac{\pi}{4}$
5. $\tan(xy)$ on the square $-2 \le x \le 2, -2 \le y \le 2$
6. $\sqrt{2x-y}$ on the disk $x^2+y^2 \le 4$

In Exercises 7–11, find the limits of the functions $f(x,y)$ as $(x,y) \to (0,0)$. Assume that polynomials, exponentials, logarithmic, and trigonometric functions are continuous.

7. $f(x,y) = e^{-x-y}$
8. $f(x,y) = x^2+y^2$
9. $f(x,y) = \dfrac{x}{x^2+1}$
10. $f(x,y) = \dfrac{x+y}{(\sin y)+2}$
11. $f(x,y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2}$ [Hint: $\lim_{t\to 0} \dfrac{\sin t}{t} = 1$.]

Problems

In Problems 12–13, show that the function $f(x,y)$ does not have a limit as $(x,y) \to (0,0)$. [Hint: Use the line $y = mx$.]

12. $f(x,y) = \dfrac{x+y}{x-y}, \quad x \ne y$
13. $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$
14. Show that $f(x,y)$ has no limit as $(x,y) \to (0,0)$ if
$$f(x,y) = \frac{xy}{|xy|}, \quad x \ne 0 \text{ and } y \ne 0.$$
15. Show that the function f does not have a limit at $(0,0)$ by examining the limits of f as $(x,y) \to (0,0)$ along the line $y = x$ and along the parabola $y = x^2$:
$$f(x,y) = \frac{x^2y}{x^4+y^2}, \quad (x,y) \ne (0,0).$$
16. Show that the function f does not have a limit at $(0,0)$ by examining the limits of f as $(x,y) \to (0,0)$ along the curve $y = kx^2$ for different values of k:
$$f(x,y) = \frac{x^2}{x^2+y}, \quad x^2+y \ne 0.$$
17. Explain why the following function is not continuous along the line $y = 0$:
$$f(x,y) = \begin{cases} 1-x, & y \ge 0, \\ -2, & y < 0. \end{cases}$$

In Problems 18–19, determine whether there is a value for c making the function continuous everywhere. If so, find it. If not, explain why not.

18. $f(x,y) = \begin{cases} c+y, & x \le 3, \\ 5-x, & x > 3. \end{cases}$
19. $f(x,y) = \begin{cases} c+y, & x \le 3, \\ 5-y, & x > 3. \end{cases}$
20. Is the following function continuous at $(0,0)$?
$$f(x,y) = \begin{cases} x^2+y^2 & \text{if } (x,y) \ne (0,0) \\ 2 & \text{if } (x,y) = (0,0) \end{cases}$$

21. What value of c makes the following function continuous at $(0,0)$?

$$f(x,y) = \begin{cases} x^2 + y^2 + 1 & \text{if } (x,y) \neq (0,0) \\ c & \text{if } (x,y) = (0,0) \end{cases}$$

22. **(a)** Use a computer to draw the graph and the contour diagram of the following function:

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

(b) Do your answers to part (a) suggest that f is continuous at $(0,0)$? Explain your answer.

23. The function f, whose graph and contour diagram are in Figures 12.86 and 12.87, is given by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

(a) Show that $f(0,y)$ and $f(x,0)$ are each continuous functions of one variable.

(b) Show that rays emanating from the origin are contained in contours of f.

(c) Is f continuous at $(0,0)$?

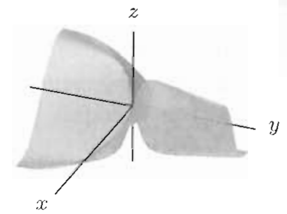


Figure 12.86: Graph of $z = xy/(x^2+y^2)$

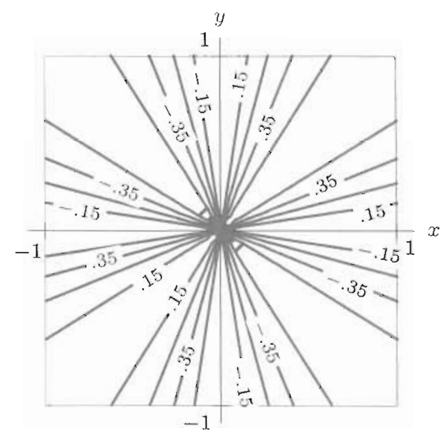


Figure 12.87: Contour diagram of $z = xy/(x^2+y^2)$

CHAPTER SUMMARY (see also Ready Reference at the end of the book)

- **3-Space**
Cartesian coordinates, x-, y- and z-axes, xy-, xz- and yz-planes, distance formula.
- **Functions of Two Variables**
Represented by: tables, graphs, formulas, cross-sections (one variable fixed), contours (function value fixed); cylinders (one variable missing).
- **Linear Functions**
Recognizing linear functions from tables, graphs, contour diagrams, formulas. Converting from one representation to another.
- **Functions of Three Variables**
Sketching level surfaces (function value fixed) in 3-space; graph of $z = f(x,y)$ is same as level surface $g(x,y,z) = f(x,y) - z = 0$.
- **Continuity**

REVIEW EXERCISES AND PROBLEMS FOR CHAPTER TWELVE

Exercises

1. On a set of x, y, and z axes oriented as in Figure 12.5 on page 641, draw a straight line through the origin, lying in the xz-plane and such that if you move along the line with your x-coordinate increasing, your z-coordinate is decreasing.

2. On a set of x, y and z axes oriented as in Figure 12.5 on page 641, draw a straight line through the origin, lying in the yz-plane and such that if you move along the line with your y-coordinate increasing, your z-coordinate is increasing.

In Exercises 3–6, determine if z is a function of x and y. If so, find a formula for the function.

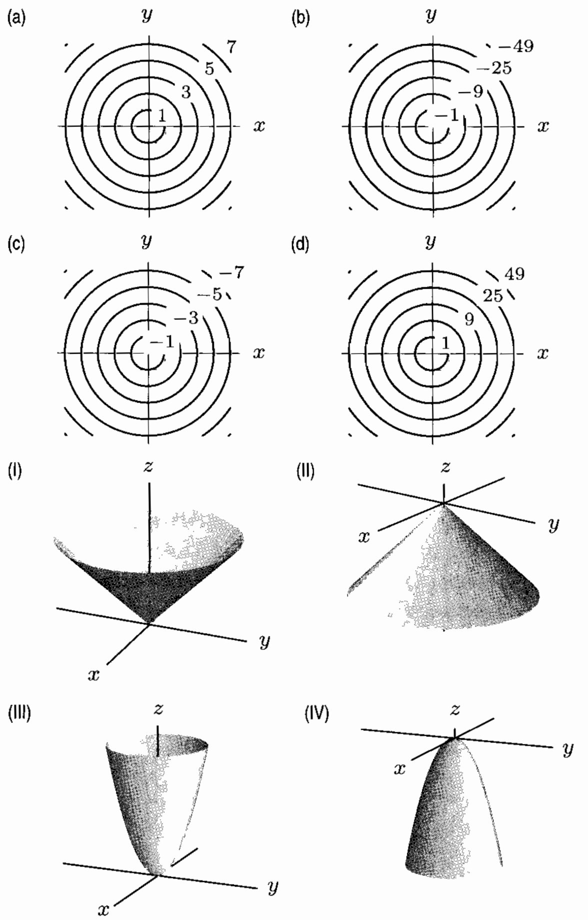
3. $6x - 4y + 2z = 10$

4. $x^2 + y^2 + z^2 = 100$

5. $3x^2 - 5y^2 + 5z = 10 + x + y$

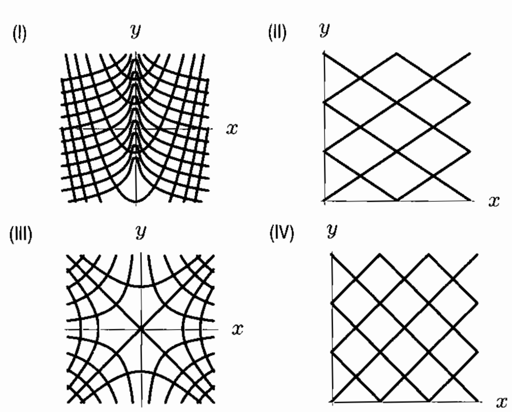
6. $x^2 + y^2 = 100$

7. Match the contour diagrams (a)–(d) with the surfaces (I)–(IV). Give reasons for your choice.



8. Match the pairs of functions (a)–(d) with the contour diagrams (I)–(IV). In each case, show which contours represent f and which represent g. (The x- and y-scales are equal.)

(a) $f(x,y) = x + y$, $g(x,y) = x - y$
(b) $f(x,y) = 2x + 3y$, $g(x,y) = 2x - 3y$
(c) $f(x,y) = x^2 - y$, $g(x,y) = 2y + \ln|x|$
(d) $f(x,y) = x^2 - y^2$, $g(x,y) = xy$



For each of the functions in Exercises 9–12, make a contour plot in the region $-2 < x < 2$ and $-2 < y < 2$. In each case, what is the equation and the shape of the contour lines?

9. $z = 3x - 5y + 1$

10. $z = \sin y$

11. $z = 2x^2 + y^2$

12. $z = e^{-2x^2 - y^2}$

13. Describe the set of points whose x coordinate is 2 and whose y coordinate is 1.

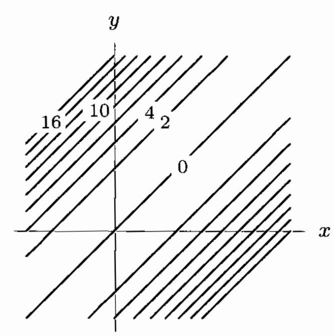
14. Find the equation of the sphere with radius 2 and centered at $(1, 0, 0)$.

15. Find the equation of the plane through the points $(0,0,2), (0,3,0), (5,0,0)$.

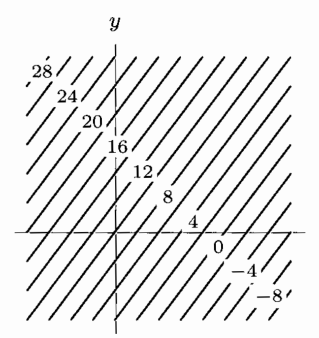
16. Find the center and radius of the sphere with equation $x^2 + 4x + y^2 - 6y + z^2 + 12z = 0$.

Which of the contour diagrams in Exercises 17–18 could represent linear functions?

17.



18.



19. **(a)** Complete the table with values of a linear function $f(x, y)$.
(b) Find a formula for $f(x, y)$.

		y		
		2.5	3.0	3.50
x	−1	6		8
	1		1	2
	3	−6		

20. The charge, C, in dollars, to use an Internet service is a function of m, the number of months of use, and t, the total number of minutes on-line:

$$C = f(m, t) = 35 + 15m + 0.05t.$$

(a) Is f a linear function?
(b) Give units for the coefficients of m and t, and interpret them as charges.
(c) Interpret the intercept 35 as a charge.
(d) Find $f(3, 800)$ and interpret your answer.

21. Find a formula for a function $f(x, y, z)$ whose level surfaces look like those in Figure 12.88.

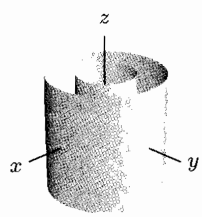


Figure 12.88

Use the catalog on page 671 to identify the surfaces in Exercises 22–23.

22. $x^2 + z^2 = 1$

23. $-x^2 + y^2 - z^2 = 0$

24. **(a)** What features of the contour diagram of $g(x, y)$ in Figure 12.89 suggest that g is linear?
(b) Assuming g is linear, find a formula for $g(x, y)$.

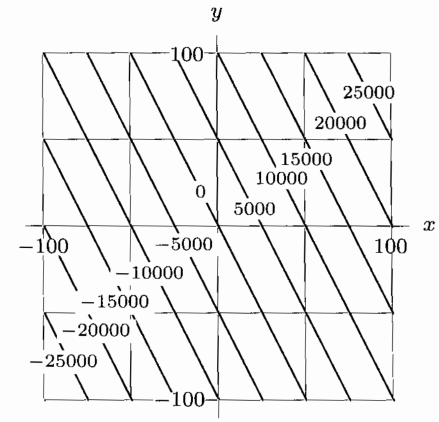


Figure 12.89

25. Figure 12.90 shows the graph of $z = f(x, y)$.

(a) Suppose y is fixed and positive. Does z increase or decrease as x increases? Graph z against x.
(b) Suppose x is fixed and positive. Does z increase or decrease as y increases? Graph z against y.

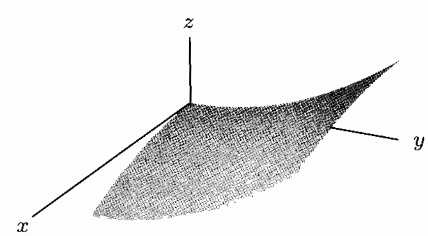


Figure 12.90

Problems

26. Use a computer or calculator to draw the graph of the vibrating guitar string function:

$$g(x, t) = \cos t \sin 2x, \quad 0 \le x \le \pi, \quad 0 \le t \le 2\pi.$$

Relate the shape of the graph to the cross-sections with t fixed and those with x fixed.

27. Consider the Cobb-Douglas production function $P = f(L, K) = 1.01L^{0.75}K^{0.25}$. What is the effect on production of doubling both labor and capital?

28. **(a)** Sketch level curves of $f(x, y) = \sqrt{x^2 + y^2} + x$ for $f = 1, 2, 3$.
(b) For what values of c can level curves $f = c$ be drawn?

29. By approaching the origin along the positive x-axis and the positive y-axis, show that the following limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{2x - y^2}{2x + y^2}.$$

30. By approaching the origin along the positive x-axis and the positive y-axis, show that the following limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x + y^2}{2x + y}.$$

31. You are in a room 30 feet long with a heater at one end. In the morning the room is 65°F. You turn on the heater, which quickly warms up to 85°F. Let $H(x, t)$ be the temperature x feet from the heater, t minutes after the heater is turned on. Figure 12.91 shows the contour diagram for H. How warm is it 10 feet from the heater 5 minutes after it was turned on? 10 minutes after it was turned on?

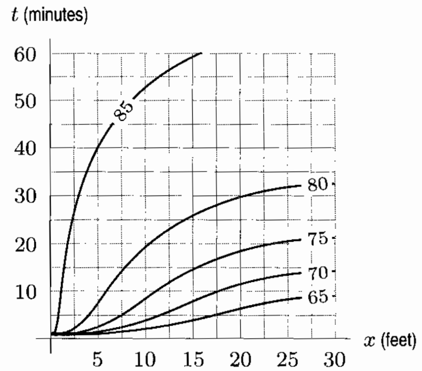


Figure 12.91

32. Using the contour diagram in Figure 12.91, sketch the graphs of the one-variable functions $H(x, 5)$ and $H(x, 20)$. Interpret the two graphs in practical terms, and explain the difference between them.

33. Sketch cross-sections of $f(r, h) = \pi r^2 h$, first keeping h fixed, then keeping r fixed.

34. Find a linear function whose graph is the plane that intersects the xy-plane along the line $y = 2x+2$ and contains the point $(1, 2, 2)$.

35. **(a)** Sketch the level curves of $z = \cos\sqrt{x^2 + y^2}$.
(b) Sketch a cross-section through the surface $z = \cos\sqrt{x^2 + y^2}$ in the plane containing the x- and z-axes. Put units on your axes.
(c) Sketch the cross-section through the surface $z = \cos\sqrt{x^2 + y^2}$ in the plane containing the z-axis and the line $y = x$ in the xy-plane.

Problems 36–39 concern a vibrating guitar string. Snapshots of the guitar string at millisecond intervals are shown in Figure 12.92.

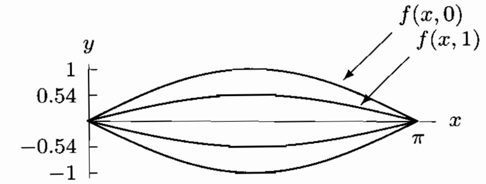


Figure 12.92: A vibrating guitar string: $f(x, t) = \cos t \sin x$ for four t values.

The guitar string is stretched tight along the x-axis from $x = 0$ to $x = \pi$. Each point on the string has an x-value, $0 \le x \le \pi$. As the string vibrates, each point on the string moves back and forth on either side of the x-axis. Let $y = f(x, t)$ be the displacement at time t of the point on the string located x units from the left end. A possible formula is

$$y = f(x, t) = \cos t \sin x, \quad 0 \le x \le \pi, \quad t \text{ in milliseconds.}$$

36. Explain what the functions $f(x, 0)$ and $f(x, 1)$ represent in terms of the vibrating string.

37. Explain what the functions $f(0, t)$ and $f(1, t)$ represent in terms of the vibrating string.

38. **(a)** Sketch graphs of y versus x for fixed t values, $t = 0$, $\pi/4$, $\pi/2$, $3\pi/4$, π.
(b) Use your graphs to explain why this function could represent a vibrating guitar string.

39. Describe the motion of the guitar strings whose displacements are given by the following:

(a) $y = g(x, t) = \cos 2t \sin x$
(b) $y = h(x, t) = \cos t \sin 2x$

CAS Challenge Problems

40. Let $A = (0, 0, 0)$ and $B = (2, 0, 0)$.

(a) Find a point C in the xy-plane that is a distance 2 from both A and B.
(b) Find a point D in 3-space that is a distance 2 from each of A, B, and C.
(c) Describe the figure obtained by joining A, B, C, and D with straight lines.

41. Let $f(x, y) = 3 + x + 2y$.

(a) Find formulas for $f(x, f(x, y))$, $f(x, f(x, f(x, y)))$ by hand.
(b) Consider $f(x, f(x, f(x, f(x, f(x, f(x, y))))))$. Conjecture a formula for this function and check your answer with a computer algebra system.

42. A function $f(x, y, z)$ has the property that $f(1, 0, 1) = 20$, $f(1, 1, 1) = 16$, and $f(1, 1, 2) = 21$.

(a) Estimate $f(1, 1, 3)$ and $f(1, 2, 1)$, assuming f is a linear function of each variable with the other variables held fixed.
(b) Suppose in fact that $f(x, y, z) = ax^2 + byz + czx^3 + d2^{x-y}$, for constants a, b, c and d. Which of your estimates in part (a) do you expect to be exact?
(c) Suppose in addition that $f(0, 0, 1) = 6$. Find an exact formula for f by solving for a, b, c, and d.
(d) Use the formula in part (c) to evaluate $f(1, 1, 3)$ and $f(1, 2, 1)$ exactly. Do the values confirm your answer to part (b)?

CHECK YOUR UNDERSTANDING

Decide if the statements in Exercises 1–5 must be true, might be true, or could not be true. The function $z = f(x, y)$ is defined everywhere.

1. The level curves corresponding to $z = 1$ and $z = -1$ cross at the origin.
2. The level curve $z = 1$ consists of the circle $x^2 + y^2 = 2$ and the circle $x^2 + y^2 = 3$, but no other points.
3. The level curve $z = 1$ consists of two lines which intersect at the origin.
4. If $z = e^{-(x^2+y^2)}$, there is a level curve for every value of z.
5. If $z = e^{-(x^2+y^2)}$, there is a level curve through every point (x, y).

Are the statements in Problems 6–66 true or false? Give reasons for your answer.

6. The function represented in Table 12.13 is linear.

Table 12.13

$u\backslash v$	1.1	1.2	1.3	1.4
3.2	11.06	12.06	13.06	14.06
3.4	11.75	12.82	13.89	14.96
3.6	12.44	13.58	14.72	15.86
3.8	13.13	14.34	15.55	16.76
4.0	13.82	15.10	16.38	17.66

7. If $f(x, y)$ is a function of two variables defined for all x and y, then $f(10, y)$ is a function of one variable.
8. The volume V of a box of height h and square base of side length s is a function of h and s.
9. If $H = f(t, d)$ is the function giving the water temperature H°C of a lake at time t hours after midnight and depth d meters, then t is a function of d and H.
10. A table for a function $f(x, y)$ cannot have any values of f appearing twice.
11. The function given by the formula $f(v, w) = e^v/w$ is an increasing function of v when w is a nonzero constant.
12. If $f(x)$ and $g(y)$ are both functions of a single variable, then the product $f(x) \cdot g(y)$ is a function of two variables.
13. Two isotherms representing distinct temperatures on a weather map cannot intersect.
14. On a weather map, there can be two isotherms which represent the same temperature but that do not intersect.
15. A function $f(x, y)$ can be an increasing function of x with y held fixed, and be a decreasing function of y with x held fixed.
16. A function $f(x, y)$ can have the property that $g(x) = f(x, 5)$ is increasing, whereas $h(x) = f(x, 10)$ is decreasing.
17. The plane $x + 2y - 3z = 1$ passes through the origin.
18. The point $(1, 2, 3)$ lies above the plane $z = 2$.
19. The graph of the equation $z = 2$ is a plane parallel to the xz-plane.
20. The points $(1, 0, 1)$ and $(0, -1, 1)$ are the same distance from the origin.
21. The plane $x + y + z = 3$ intersects the x-axis when $x = 3$.
22. The point $(2, -1, 3)$ lies on the graph of the sphere $(x - 2)^2 + (y + 1)^2 + (z - 3)^2 = 25$.
23. There is only one point in the yz-plane that is a distance 3 from the point $(3, 0, 0)$.
24. There is only one point in the yz-plane that is distance 5 from the point $(3, 0, 0)$.
25. The sphere $x^2 + y^2 + z^2 = 10$ intersects the plane $x = 10$.
26. If the point $(0, b, 0)$ has distance 4 from the plane $y = 0$, then b must be 4.
27. The cross-section of the function $f(x, y) = x + y^2$ with $y = 1$ is a line.
28. The function $g(x, y) = 1 - y^2$ has identical parabolas for all cross-sections with $x = c$.
29. The function $g(x, y) = 1 - y^2$ has lines for all cross-sections with $y = c$.
30. The graphs of $f(x, y) = \sin(xy)$ and $g(x, y) = \sin(xy) + 2$ never intersect.
31. The graphs of $f(x, y) = x^2 + y^2$ and $g(x, y) = 1 - x^2 - y^2$ intersect in a circle.
32. If all of the $x = c$ cross-sections of the graph of $f(x, y)$ are lines, then the graph of f is a plane.
33. The only point of intersection of the graphs of $f(x, y)$ and $-f(x, y)$ is the origin.
34. A line parallel to the z-axis can intersect the graph of $f(x, y)$ at most once.
35. A line parallel to the y-axis can intersect the graph of $f(x, y)$ at most once.
36. The point $(0, 0, 10)$ is the highest point on the graph of the function $f(x, y) = 10 - x^2 - y^2$.
37. The contours of the function $f(x, y) = y^2 + (x - 2)^2$ are either circles or a single point.
38. Two contours of $f(x, y)$ with different heights never intersect.
39. If the contours of $g(x, y)$ are concentric circles, then the graph of g is a cone.

40. If the contours for $f(x, y)$ get closer together in a certain direction, then f is increasing in that direction.

41. If all of the contours of $f(x, y)$ are parallel lines, then the graph of f is a plane.

42. The $h = 1$ contour of the function $h(x, y) = xy$ is a hyperbola.

43. If the $f = 10$ contour of the function $f(x, y)$ is identical to the $g = 10$ contour of the function $g(x, y)$, then $f(x, y) = g(x, y)$ for all (x, y).

44. A function $f(x, y)$ has a contour $f = c$ for every value of c.

45. The $f = 5$ contour of the function $f(x, y)$ is identical to the $g = 0$ contour of the function $g(x, y) = f(x, y) - 5$.

46. Contours of $f(x, y) = 3x + 2y$ are lines with slope 3.

47. If the contours of f are all parallel lines, then f is linear.

48. If f is linear, then the contours of f are parallel lines.

49. The function f satisfying $f(0, 0) = 1, f(0, 1) = 4, f(0, 3) = 5$ cannot be linear.

50. The graph of a linear function is always a plane.

51. The cross-section $x = c$ of a linear function $f(x, y)$ is always a line.

52. There is no linear function $f(x, y)$ with a graph parallel to the xy-plane.

53. There is no linear function $f(x, y)$ with a graph parallel to the xz-plane.

54. A linear function $f(x, y) = 2x + 3y - 5$, has exactly one point (a, b) satisfying $f(a, b) = 0$.

55. In a table of values of a linear function, the columns have the same slope as the rows.

56. There is exactly one linear function $f(x, y)$ whose $f = 0$ contour is $y = 2x + 1$.

57. The graph of the function $f(x, y) = x^2 + y^2$ is the same as the level surface $g(x, y, z) = x^2 + y^2 - z = 0$.

58. The graph of $f(x, y) = \sqrt{1 - x^2 - y^2}$ is the same as the level surface $g(x, y, z) = x^2 + y^2 + z^2 = 1$.

59. Any surface which is the graph of a two-variable function $f(x, y)$ can also be represented as the level surface of a three-variable function $g(x, y, z)$.

60. Any surface which is the level surface of a three-variable function $g(x, y, z)$ can also be represented as the graph of a two-variable function $f(x, y)$.

61. The level surfaces of the function $g(x, y, z) = x + 2y + z$ are parallel planes.

62. The level surfaces of $g(x, y, z) = x^2 + y + z^2$ are cylinders with axis along the y-axis.

63. A level surface of a function $g(x, y, z)$ cannot be a single point.

64. If $g(x, y, z) = ax + by + cz + d$, where a, b, c, d are nonzero constants, then the level surfaces of g are planes.

65. If the level surfaces of g are planes, then $g(x, y, z) = ax + by + cz + d$, where a, b, c, d are constants.

66. If the level surfaces $g(x, y, z) = k_1$ and $g(x, y, z) = k_2$ are the same surface, then $k_1 = k_2$.

PROJECTS FOR CHAPTER TWELVE

1. A Heater in a Room

Figure 12.93 shows the contours of the temperature along one wall of a heated room through one winter day, with time indicated as on a 24-hour clock. The room has a heater located at the left-most corner of the wall and one window in the wall. The heater is controlled by a thermostat about 2 feet from the window.

(a) Where is the window?
(b) When is the window open?
(c) When is the heat on?
(d) Draw graphs of the temperature along the wall of the room at 6 am, at 11 am, at 3 pm (15 hours) and at 5 pm (17 hours).
(e) Draw a graph of the temperature as a function of time at the heater, at the window and midway between them.
(f) The temperature at the window at 5 pm (17 hours) is less than at 11 am. Why do you think this might be?
(g) To what temperature do you think the thermostat is set? How do you know?
(h) Where is the thermostat?

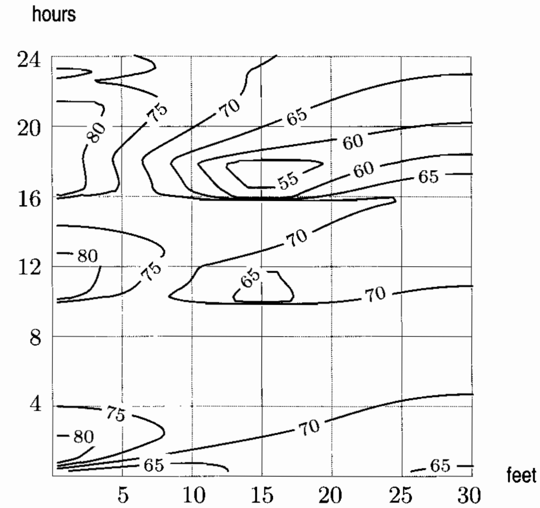


Figure 12.93:

2. Light in a Wave-guide

Figure 12.94 shows the contours of light intensity as a function of location and time in a microscopic wave-guide.

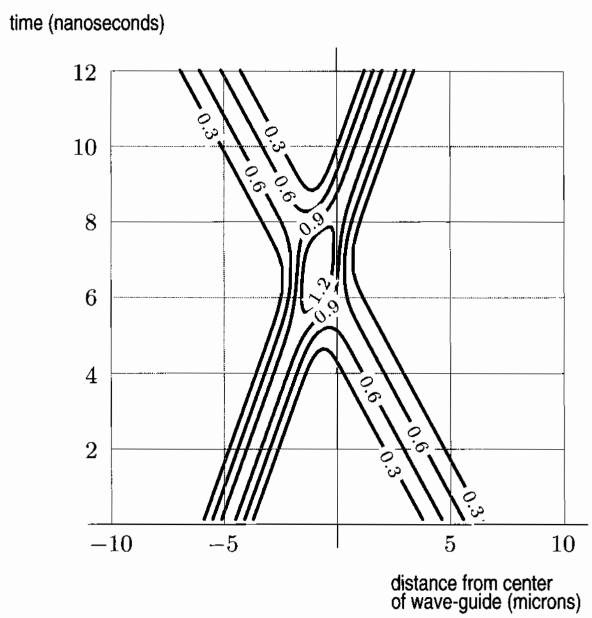


Figure 12.94

(a) Draw graphs showing intensity as a function of location at times 0, 2, 4, 6, 8, and 10 nanoseconds.

(b) If you could create an animation showing how the graph of intensity as a function of location varies as time passes, what would it look like?

(c) Draw a graph of intensity as a function of time at locations −5, 0, and 5 microns from center of wave-guide.

(d) Describe what the light beams are doing in this wave-guide.