Unit Vectors

A *unit vector* is a vector whose magnitude is 1. The vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ are unit vectors in the directions of the coordinate axes. It is often helpful to find a unit vector in the same direction as a given vector $\vec{v}$. Suppose that $\|\vec{v}\| = 10$; a unit vector in the same direction as $\vec{v}$ is $\vec{v}/10$. In general, a unit vector in the direction of any nonzero vector $\vec{v}$ is

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}.$$

Example 5 Find a unit vector, $\vec{u}$, in the direction of the vector $\vec{v} = \vec{i} + 3\vec{j}$.

Solution If $\vec{v} = \vec{i} + 3\vec{j}$, then $\|\vec{v}\| = \sqrt{1^2 + 3^2} = \sqrt{10}$. Thus, a unit vector in the same direction is given by

$$\vec{u} = \frac{\vec{v}}{\sqrt{10}} = \frac{1}{\sqrt{10}}(\vec{i} + 3\vec{j}) = \frac{1}{\sqrt{10}}\vec{i} + \frac{3}{\sqrt{10}}\vec{j} \approx 0.32\vec{i} + 0.95\vec{j}.$$

Example 6 Find a unit vector at the point (x, y, z) that points radially outward away from the origin.

Solution The vector from the origin to (x, y, z) is the position vector

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}.$$

Thus, if we put its tail at (x, y, z) it will point away from the origin. Its magnitude is

$$\|\vec{r}\| = \sqrt{x^2 + y^2 + z^2},$$

so a unit vector pointing in the same direction is

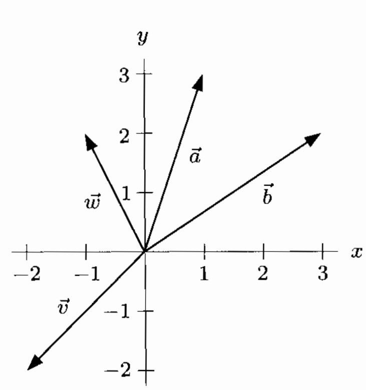
$$\frac{\vec{r}}{\|\vec{r}\|} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{x^2+y^2+z^2}}\vec{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\vec{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\vec{k}.$$

Exercises and Problems for Section 13.1

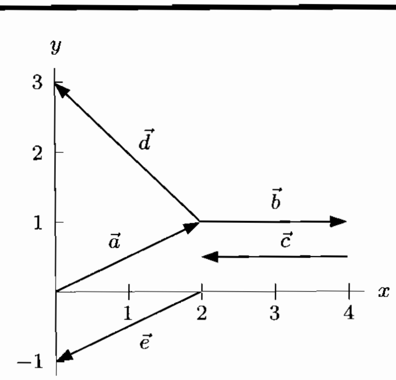
Exercises

Resolve the vectors in Exercises 1–6 into components.

1.



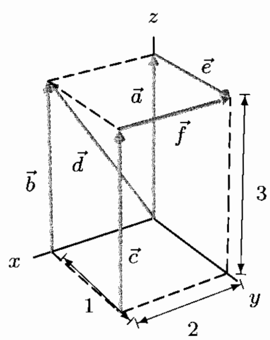
2.



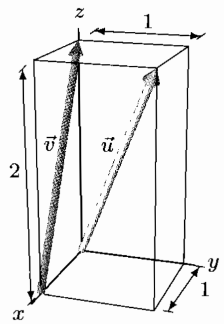
3. A vector starting at the point $Q = (4, 6)$ and ending at the point $P = (1, 2)$.

4. A vector starting at the point $P = (1, 2)$ and ending at the point $Q = (4, 6)$.

5.



6.



For Exercises 7–14, perform the indicated computation.

7. $(4\vec{i} + 2\vec{j}) - (3\vec{i} - \vec{j})$

8. $(\vec{i} + 2\vec{j}) + (-3)(2\vec{i} + \vec{j})$

9. $-4(\vec{i} - 2\vec{j}) - 0.5(\vec{i} - \vec{k})$

10. $2(0.45\vec{i} - 0.9\vec{j} - 0.01\vec{k}) - 0.5(1.2\vec{i} - 0.1\vec{k})$

11. $(3\vec{i} - 4\vec{j} + 2\vec{k}) - (6\vec{i} + 8\vec{j} - \vec{k})$

12. $(4\vec{i} - 3\vec{j} + 7\vec{k}) - 2(5\vec{i} + \vec{j} - 2\vec{k})$

13. $(0.6\vec{i} + 0.2\vec{j} - \vec{k}) + (0.3\vec{i} + 0.3\vec{k})$

14. $\frac{1}{2}(2\vec{i} - \vec{j} + 3\vec{k}) + 3(\vec{i} - \frac{1}{6}\vec{j} + \frac{1}{2}\vec{k})$

Find the length of the vectors in Exercises 15–19.

15. $\vec{v} = \vec{i} - \vec{j} + 2\vec{k}$

16. $\vec{z} = \vec{i} - 3\vec{j} - \vec{k}$

17. $\vec{v} = \vec{i} - \vec{j} + 3\vec{k}$

18. $\vec{v} = 7.2\vec{i} - 1.5\vec{j} + 2.1\vec{k}$

19. $\vec{v} = 1.2\vec{i} - 3.6\vec{j} + 4.1\vec{k}$

For Exercises 20–25, perform the indicated operations on the following vectors:

$$\vec{a} = 2\vec{j} + \vec{k}, \quad \vec{b} = -3\vec{i} + 5\vec{j} + 4\vec{k}, \quad \vec{c} = \vec{i} + 6\vec{j},$$

$$\vec{x} = -2\vec{i} + 9\vec{j}, \quad \vec{y} = 4\vec{i} - 7\vec{j}, \quad \vec{z} = \vec{i} - 3\vec{j} - \vec{k}.$$

20. $4\vec{z}$

21. $5\vec{a} + 2\vec{b}$

22. $\vec{a} + \vec{z}$

23. $2\vec{c} + \vec{x}$

24. $2\vec{a} + 7\vec{b} - 5\vec{z}$

25. $\|\vec{y} - \vec{x}\|$

26. **(a)** Draw the position vector for $\vec{v} = 5\vec{i} - 7\vec{j}$.
   (b) What is $\|\vec{v}\|$?
   (c) Find the angle between $\vec{v}$ and the positive x-axis.

27. Find the unit vector in the direction of $0.06\vec{i} - 0.08\vec{k}$.

Problems

28. Find a unit vector in the opposite direction to $\vec{v} = 2\vec{i} - \vec{j} - \sqrt{11}\vec{k}$.

29. Find the value(s) of a making $\vec{v} = 5a\vec{i} - 3\vec{j}$ parallel to $\vec{w} = a^2\vec{i} + 6\vec{j}$.

30. Find a vector with length 2 that points in the same direction as $\vec{i} - \vec{j} + 2\vec{k}$.

31. **(a)** Find a unit vector from the point $P = (1, 2)$ and toward the point $Q = (4, 6)$.
   (b) Find a vector of length 10 pointing in the same direction.

32. If north is the direction of the positive y-axis and east is the direction of the positive x-axis, give the unit vector pointing northwest.

33. Resolve the following vectors into components:
   (a) The vector in 2-space of length 2 pointing up and to the right at an angle of $\pi/4$ with the x-axis.
   (b) The vector in 3-space of length 1 lying in the xz-plane pointing upward at an angle of $\pi/6$ with the positive x-axis.

34. **(a)** From Figure 13.16, read off the coordinates of the five points, A, B, C, D, E, and thus resolve into components the following two vectors: $\vec{u} = (2.5)\overrightarrow{AB} + (-0.8)\overrightarrow{CD}$, $\vec{v} = (2.5)\overrightarrow{BA} - (-0.8)\overrightarrow{CD}$
   (b) What is the relation between $\vec{u}$ and $\vec{v}$? Why was this to be expected?

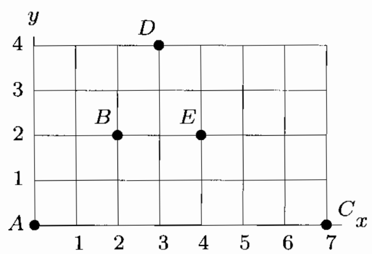


Figure 13.16

35. Find the components of a vector $\vec{p}$ which has the same direction as $\overrightarrow{EA}$ in Figure 13.16 and whose length equals two units.

36. For each of the four statements below, answer the following questions: Does the statement make sense? If yes, is it true for all possible choices of $\vec{a}$ and $\vec{b}$? If no, why not?

(a) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ **(b)** $\vec{a} + \|\vec{b}\| = \|\vec{a} + \vec{b}\|$
(c) $\|\vec{b} + \vec{a}\| = \|\vec{a} + \vec{b}\|$ **(d)** $\|\vec{a} + \vec{b}\| = \|\vec{a}\| + \|\vec{b}\|$.

37. Two adjacent sides of a regular hexagon are given as the vectors $\vec{u}$ and $\vec{v}$ in Figure 13.17. Label the remaining sides in terms of $\vec{u}$ and $\vec{v}$.

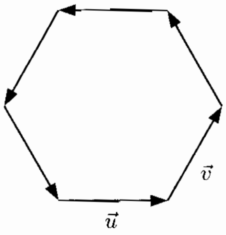


Figure 13.17

38. For what values of t are the following pairs of vectors parallel?

(a) $2\vec{i} + (t^2 + \frac{2}{3}t + 1)\vec{j} + t\vec{k}$, $6\vec{i} + 8\vec{j} + 3\vec{k}$
(b) $t\vec{i} + \vec{j} + (t-1)\vec{k}$, $2\vec{i} - 4\vec{j} + \vec{k}$
(c) $2t\vec{i} + t\vec{j} + t\vec{k}$, $6\vec{i} + 3\vec{j} + 3\vec{k}$.

39. Find all vectors $\vec{v}$ in 2 dimensions having $\|\vec{v}\| = 5$ and the $\vec{i}$-component of $\vec{v}$ is $3\vec{i}$.

40. Find all vectors $\vec{v}$ in the plane such that $\|\vec{v}\| = 1$ and $\|\vec{v} + \vec{i}\| = 1$.

41. A truck is traveling due north at 30 km/hr approaching a crossroad. On a perpendicular road a police car is traveling west toward the intersection at 40 km/hr. Both vehicles will reach the crossroad in exactly one hour. Find the vector currently representing the displacement of the truck with respect to the police car.

42. Figure 13.18 shows a molecule with four atoms at O, A, B and C. Verify that every atom in the molecule is 2 units away from every other atom.

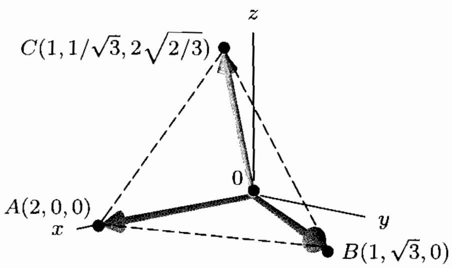


Figure 13.18

43. Show that the medians of a triangle intersect at a point $\frac{1}{3}$ of the way along each median from the side it bisects.

44. In the game of laser tag, you shoot a harmless laser gun and try to hit a target worn at the waist by other players. Suppose you are standing at the origin of a three dimensional coordinate system and that the xy-plane is the floor. Suppose that waist-high is 3 feet above floor level and that eye level is 5 feet above the floor. Three of your friends are your opponents. One is standing so that his target is 30 feet along the x-axis, the other lying down so that his target is at the point $x = 20$, $y = 15$, and the third lying in ambush so that his target is at a point 8 feet above the point $x = 12$, $y = 30$.

(a) If you aim with your gun at eye level, find the vector from your gun to each of the three targets.
(b) If you shoot from waist height, with your gun one foot to the right of the center of your body as you face along the x-axis, find the vector from your gun to each of the three targets.

13.2 VECTORS IN GENERAL

Besides displacement, there are many quantities that have both magnitude and direction and are added and multiplied by scalars in the same way as displacements. Any such quantity is called a *vector* and is represented by an arrow in the same manner we represent displacements. The length of the arrow is the *magnitude* of the vector, and the direction of the arrow is the direction of the vector.

Velocity Versus Speed

The speed of a moving body tells us how fast it is moving, say 80 km/hr. The speed is just a number; it is therefore a scalar. The velocity, on the other hand, tells us both how fast the body is moving and the direction of motion; it is a vector. For instance, if a car is heading northeast at 80 km/hr, then its velocity is a vector of length 80 pointing northeast.

> The **velocity vector** of a moving object is a vector whose magnitude is the speed of the object and whose direction is the direction of its motion.

shows the quantities $q_1, q_2, \ldots, q_n$ consumed of each of n different goods. A *price* vector

$$\vec{p} = (p_1, p_2, \ldots, p_n)$$

contains the prices of n different items.

In 1907, Hermann Minkowski used vectors with four components when he introduced *space-time coordinates*, whereby each event is assigned a vector position $\vec{v}$ with four coordinates, three for its position in space and one for time:

$$\vec{v} = (x, y, z, t).$$

Example 7 Suppose the vector $\vec{I}$ represents the number of copies, in thousands, made by each of four copy centers in the month of December and $\vec{J}$ represents the number of copies made at the same four copy centers during the previous eleven months (the "year-to-date"). If $\vec{I} = (25, 211, 818, 642)$, and $\vec{J} = (331, 3227, 1377, 2570)$, compute $\vec{I} + \vec{J}$. What does this sum represent?

Solution The sum is

$$\vec{I} + \vec{J} = (25 + 331, 211 + 3227, 818 + 1377, 642 + 2570) = (356, 3438, 2195, 3212).$$

Each term in $\vec{I} + \vec{J}$ represents the sum of the number of copies made in December plus those in the previous eleven months, that is, the total number of copies made during the entire year at that particular copy center.

Example 8 The price vector $\vec{p} = (p_1, p_2, p_3)$ represents the prices in dollars of three goods. Write a vector which gives the prices of the same goods in cents.

Solution The prices in cents are $100p_1$, $100p_2$, and $100p_3$ respectively, so the new price vector is

$$(100p_1, 100p_2, 100p_3) = 100\vec{p}.$$

Exercises and Problems for Section 13.2

Exercises

In Exercises 1–5, say whether the given quantity is a vector or a scalar.

1. The population of the US.

2. The distance from Seattle to St. Louis.

3. The temperature at a point on the earth's surface.

4. The magnetic field at a point on the earth's surface.

5. The populations of each of the 50 states.

6. A car is traveling at a speed of 50 km/hr. The positive y-axis is north and the positive x-axis is east. Resolve the car's velocity vector (in 2-space) into components if the car is traveling in each of the following directions:

(a) East **(b)** South
(c) Southeast **(d)** Northwest.

7. Give the components of the velocity vector for wind blowing at 10 km/hr toward the southeast. (Assume north is the positive y-direction.)

8. Give the components of the velocity vector of a boat which is moving at 40 km/hr in a direction 20° south of west. (Assume north is in the positive y-direction.)

9. Which is traveling faster, a car whose velocity vector is $21\vec{i} + 35\vec{j}$, or a car whose velocity vector is $40\vec{i}$, assuming that the units are the same for both directions?

10. What angle does a force of $\vec{F} = 15\vec{i} + 18\vec{j}$ make with the x-axis?

Problems

11. The velocity of the current in a river is $\vec{c} = 0.6\vec{i} + 0.8\vec{j}$ km/hr. A boat moves relative to the water with velocity $\vec{v} = 8\vec{i}$ km/hr.

(a) What is the speed of the boat relative to the riverbed?
(b) What angle does the velocity of the boat relative to the riverbed make with the vector $\vec{v}$? What does this angle tell us in practical terms?

12. Suppose the current in Problem 11 is twice as fast and in the opposite direction. What is the speed of the boat with respect to the riverbed?

13. A plane is heading due east and climbing at the rate of 80 km/hr. If its airspeed is 480 km/hr and there is a wind blowing 100 km/hr to the northeast, what is the ground speed of the plane?

14. An airplane is flying at an airspeed of 500 km/hr in a wind blowing at 60 km/hr toward the southeast. In what direction should the plane head to end up going due east? What is the airplane's speed relative to the ground?

15. An airplane is flying at an airspeed of 600 km/hr in a cross-wind that is blowing from the northeast at a speed of 50 km/hr. In what direction should the plane head to end up going due east?

16. The current in a river is pushing a boat in direction 25° north of east with a speed of 12 km/hr. The wind is pushing the same boat in a direction 80° south of east with a speed of 7 km/hr. Find the velocity vector of the boat's engine (relative to the water) if the boat actually moves due east at a speed of 40 km/hr relative to the ground.

17. A large ship is being towed by two tugs. The larger tug exerts a force which is 25% greater than the smaller tug and at an angle of 30 degrees north of east. Which direction must the smaller tug pull to ensure that the ship travels due east?

18. A particle moving with speed v hits a barrier at an angle of 60° and bounces off at an angle of 60° in the opposite direction with speed reduced by 20 percent. See Figure 13.25. Find the velocity vector of the object after impact.

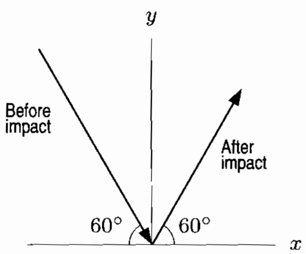


Figure 13.25

19. There are five students in a class. Their scores on the midterm (out of 100) are given by the vector $\vec{v} = (73, 80, 91, 65, 84)$. Their scores on the final (out of 100) are given by $\vec{w} = (82, 79, 88, 70, 92)$. If the final counts twice as much as the midterm, find a vector giving the total scores (as a percentage) of the students.

20. The price vector of beans, rice, and tofu is $(0.30, 0.20, 0.50)$ in dollars per pound. Express it in dollars per ounce.

21. Two forces, represented by the vectors $\vec{F}_1 = 8\vec{i} - 6\vec{j}$ and $\vec{F}_2 = 3\vec{i} + 2\vec{j}$, are acting on an object. Give a vector representing the force that must be applied to the object if it is to remain stationary.

22. One force is pushing an object in a direction 50° south of east with a force of 25 newtons. A second force is simultaneously pushing the object in a direction 70° north of west with a force of 60 newtons. If the object is to remain stationary, give the direction and magnitude of the third force which must be applied to the object to counterbalance the first two.

23. An object P is pulled by a force $\vec{F_1}$ of magnitude 15 lb at an angle of 20 degrees north of east. In what direction must a force $\vec{F_2}$ of magnitude 20 lb pull to ensure that P moves due east?

24. An airplane heads northeast at an airspeed of 700 km/hr, but there is a wind blowing from the west at 60 km/hr. In what direction does the plane end up flying? What is its speed relative to the ground?

25. A man wishes to row the shortest possible distance from north to south across a river which is flowing at 4 km/hr from the east. He can row at 5 km/hr.

(a) In which direction should he steer?
(b) If there is a wind of 10 km/hr from the southwest, in which direction should he steer to try and go directly across the river? What happens?

26. An object is moving counterclockwise at a constant speed around the circle $x^2 + y^2 = 1$, where x and y are measured in meters. It completes one revolution every minute.

(a) What is its speed?
(b) What is its velocity vector 30 seconds after it passes the point $(1, 0)$? Does your answer change if the object is moving clockwise? Explain.

27. An object is attached by a string to a fixed point and rotates 30 times per minute in a horizontal plane. Show that the speed of the object is constant but the velocity is not. What does this imply about the acceleration?

Use the geometric definition of addition and scalar multiplication to explain each of the properties in Problems 28–35.

28. $\vec{w} + \vec{v} = \vec{v} + \vec{w}$

29. $(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$

30. $\alpha(\vec{v} + \vec{w}) = \alpha\vec{v} + \alpha\vec{w}$

31. $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$

32. $\vec{v} + \vec{0} = \vec{v}$

33. $1\vec{v} = \vec{v}$

34. $\vec{v} + (-1)\vec{w} = \vec{v} - \vec{w}$

35. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

36. The earth is at the origin, the moon is at the point $(384, 0)$, and a spaceship is at $(280, 90)$, where distance is in thousands of kilometers.

(a) What is the displacement vector of the moon relative to the earth? Of the spaceship relative to the earth? Of the spaceship relative to the moon?

(b) How far is the spaceship from the earth? From the moon?

(c) The gravitational force on the spaceship from the earth is 461 newtons and from the moon is 26 newtons. What is the resulting force?

13.3 THE DOT PRODUCT

We have seen how to add vectors; can we multiply two vectors together? In the next two sections we will see two different ways of doing so: the *scalar product* (or *dot product*) which produces a scalar, and the *vector product* (or *cross product*), which produces a vector.

Definition of the Dot Product

The dot product links geometry and algebra. We already know how to calculate the length of a vector from its components; the dot product gives us a way of computing the angle between two vectors. For any two vectors $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$, shown in Figure 13.26, we define a scalar as follows:

> The following two definitions of the **dot product**, or **scalar product**, $\vec{v} \cdot \vec{w}$, are equivalent:
>
> - **Geometric definition**
>
> $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta$ where θ is the angle between $\vec{v}$ and $\vec{w}$ and $0 \le \theta \le \pi$.
> - **Algebraic definition**
>
> $\vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2 + v_3w_3.$
>
> Notice that the dot product of two vectors is a *number*.

Why don't we give just one definition of $\vec{v} \cdot \vec{w}$? The reason is that both definitions are equally important; the geometric definition gives us a picture of what the dot product means and the algebraic definition gives us a way of calculating it.

How do we know the two definitions are equivalent — that is, they really do define the same thing? First, we observe that the two definitions give the same result in a particular example. Then we show why they are equivalent in general.

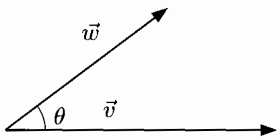


Figure 13.26: The vectors $\vec{v}$ and $\vec{w}$

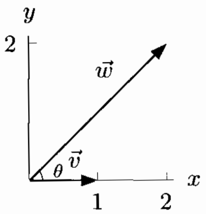


Figure 13.27: Calculating the dot product of the vectors $v = \vec{i}$ and $\vec{w} = 2\vec{i} + 2\vec{j}$ geometrically and algebraically gives the same result

Example 1 Suppose $\vec{v} = \vec{i}$ and $\vec{w} = 2\vec{i} + 2\vec{j}$. Compute $\vec{v} \cdot \vec{w}$ both geometrically and algebraically.

Solution To use the geometric definition, see Figure 13.27. The angle between the vectors is $\pi/4$, or $45°$, and the lengths of the vectors are given by

$$\|\vec{v}\| = 1 \quad \text{and} \quad \|\vec{w}\| = 2\sqrt{2}.$$

we assume $0 \leq \theta \leq \pi/2$. Figure 13.33 shows how we can resolve $\vec{F}$ into components that are parallel and perpendicular to $\vec{d}$:

$$\vec{F} = \vec{F}_{\text{parallel}} + \vec{F}_{\text{perp}},$$

Then the work done by $\vec{F}$ is defined to be

$$W = \|\vec{F}_{\text{parallel}}\| \, \|\vec{d}\|.$$

We see from Figure 13.33 that $\vec{F}_{\text{parallel}}$ has magnitude $\|\vec{F}\| \cos\theta$. So the work is given by the dot product:

$$W = (\|\vec{F}\| \cos\theta)\|\vec{d}\| = \|\vec{F}\|\|\vec{d}\| \cos\theta = \vec{F} \cdot \vec{d}.$$

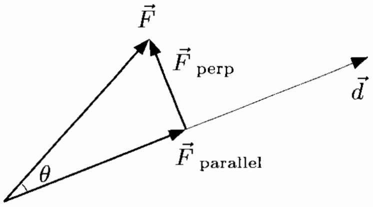


Figure 13.33: Resolving the force $\vec{F}$ into two forces, one parallel to $\vec{d}$, one perpendicular to $\vec{d}$

The formula $W = \vec{F} \cdot \vec{d}$ holds when $\pi/2 < \theta \leq \pi$ also. In that case, the work done by the force is negative and the object is moving against the force. Thus, we have the following definition:

> The **work**, W, done by a force $\vec{F}$ acting on an object through a displacement $\vec{d}$ is given by
>
> $$W = \vec{F} \cdot \vec{d}.$$

Notice that if the vectors $\vec{F}$ and $\vec{d}$ are parallel and in the same direction, with magnitudes F and d, then $\cos\theta = \cos 0 = 1$, so $W = \|\vec{F}\|\|\vec{d}\| = Fd$, which is the original definition. When the vectors are perpendicular, $\cos\theta = \cos\frac{\pi}{2} = 0$, so $W = 0$ and no work is done in the technical definition of the word. For example, if you carry a heavy box across the room at the same horizontal height, no work is done by gravity because the force of gravity is vertical but the motion is horizontal.

Exercises and Problems for Section 13.3

Exercises

For Exercises 1–9, perform the following operations on the given 3-dimensional vectors.

$$\vec{a} = 2\vec{j} + \vec{k} \qquad \vec{b} = -3\vec{i} + 5\vec{j} + 4\vec{k} \qquad \vec{c} = \vec{i} + 6\vec{j}$$

$$\vec{y} = 4\vec{i} - 7\vec{j} \qquad \vec{z} = \vec{i} - 3\vec{j} - \vec{k}$$

1. $\vec{a} \cdot \vec{y}$

2. $\vec{c} \cdot \vec{y}$

3. $\vec{a} \cdot \vec{b}$

4. $\vec{a} \cdot \vec{z}$

5. $\vec{c} \cdot \vec{a} + \vec{a} \cdot \vec{y}$

6. $\vec{a} \cdot (\vec{c} + \vec{y})$

7. $(\vec{a} \cdot \vec{b})\vec{a}$

8. $(\vec{a} \cdot \vec{y})(\vec{c} \cdot \vec{z})$

9. $((\vec{c} \cdot \vec{c})\vec{a}) \cdot \vec{a}$

In Exercises 10–14, find a normal vector to the plane.

10. $2x + y - z = 23$

11. $1.5x + 3.2y + z = 0$

12. $z = 3x + 4y - 7$

13. $z - 5(x - 2) = 3(5 - y)$

14. $\pi(x - 1) = (1 - \pi)(y - z) + \pi$

15. Give a unit vector

(a) In the same direction as $\vec{v} = 2\vec{i} + 3\vec{j}$.
(b) Perpendicular to $\vec{v}$.

16. **(a)** Find a vector perpendicular to the plane $z = 2 + 3x - y$.
(b) Find a vector parallel to the plane.

17. **(a)** Find a vector perpendicular to the plane $z = 2x + 3y$.
(b) Find a vector parallel to the plane.

In Exercises 18–23, given vector $\vec{v} = 3\vec{i} + 4\vec{j}$ and force vector $\vec{F}$, find:

(a) The component of $\vec{F}$ parallel to $\vec{v}$.
(b) The component of $\vec{F}$ perpendicular to $\vec{v}$.
(c) The work, W, done by force $\vec{F}$ through displacement $\vec{v}$.

18. $\vec{F} = 4\vec{i} + \vec{j}$
19. $\vec{F} = 0.2\vec{i} - 0.5\vec{j}$
20. $\vec{F} = 9\vec{i} + 12\vec{j}$
21. $\vec{F} = -0.4\vec{i} + 0.3\vec{j}$
22. $\vec{F} = -3\vec{i} - 5\vec{j}$
23. $\vec{F} = -6\vec{i} - 8\vec{j}$

In Exercises 24–27, the force on an object is $\vec{F} = -20\vec{j}$. For vector $\vec{v}$, find:

(a) The component of $\vec{F}$ parallel to $\vec{v}$.
(b) The component of $\vec{F}$ perpendicular to $\vec{v}$.
(c) The work, W, done by force $\vec{F}$ through displacement $\vec{v}$.

24. $\vec{v} = 2\vec{i} + 3\vec{j}$
25. $\vec{v} = 5\vec{i} - \vec{j}$
26. $\vec{v} = 3\vec{j}$
27. $\vec{v} = 5\vec{i}$

Problems

28. Match the planes in (a)–(d) with **one or more** of the descriptions in (I)–(IV). No reasons needed.

(a) $3x - y + z = 0$
(b) $4x + y + 2z - 5 = 0$
(c) $x + y = 5$
(d) $x = 5$

I Goes through the origin.
II Has a normal vector parallel to the xy-plane.
III Goes through the point $(0, 5, 0)$.
IV Has a normal vector whose dot products with $\vec{i}$, $\vec{j}$, $\vec{k}$ are all positive.

29. Which pairs (if any) of vectors from the following list

(a) Are perpendicular?
(b) Are parallel?
(c) Have an angles less than $\pi/2$ between them?
(d) Have an angle of more than $\pi/2$ between them?

$$\vec{a} = \vec{i} - 3\vec{j} - \vec{k}, \qquad \vec{b} = \vec{i} + \vec{j} + 2\vec{k},$$
$$\vec{c} = -2\vec{i} - \vec{j} + \vec{k}, \qquad \vec{d} = -\vec{i} - \vec{j} + \vec{k}.$$

30. Which pairs of the vectors $\sqrt{3}\vec{i} + \vec{j}$, $3\vec{i} + \sqrt{3}\vec{j}$, $\vec{i} - \sqrt{3}\vec{j}$ are parallel and which are perpendicular?

31. Compute the angle between the vectors $\vec{i} + \vec{j} + \vec{k}$ and $\vec{i} - \vec{j} - \vec{k}$.

32. **(a)** Give a vector that is parallel to, but not equal to, $\vec{v} = 4\vec{i} + 3\vec{j}$.
(b) Give a vector that is perpendicular to $\vec{v}$.

33. What values of a make $\vec{v} = 2a\vec{i} - a\vec{j} + 16\vec{k}$ perpendicular to $\vec{w} = 5\vec{i} + a\vec{j} - \vec{k}$?

34. Let θ be the angle between $\vec{v}$ and $\vec{w}$, with $0 < \theta < \pi/2$. What is the effect on $\vec{v} \cdot \vec{w}$ of increasing each of the following quantities? Does $\vec{v} \cdot \vec{w}$ increase or decrease?

(a) $||\vec{v}\,||$
(b) θ

35. Write $\vec{a} = 3\vec{i} + 2\vec{j} - 6\vec{k}$ as the sum of two vectors, one parallel, and one perpendicular, to $\vec{d} = 2\vec{i} - 4\vec{j} + \vec{k}$.

36. Find angle BAC if $A = (2, 2, 2)$, $B = (4, 2, 1)$, and $C = (2, 3, 1)$.

In Problems 37–42, find an equation of a plane that satisfies the given conditions.

37. Through $(1, 5, 2)$ perpendicular to $3\vec{i} - \vec{j} + 4\vec{k}$

38. Through $(2, -1, 3)$ perpendicular to $5\vec{i} + 4\vec{j} - \vec{k}$.

39. Perpendicular to the vector $2\vec{i} - 3\vec{j} + 7\vec{k}$ and passing through the point $(1, -1, 2)$.

40. Parallel to the plane $2x + 4y - 3z = 1$ and through the point $(1, 0, -1)$.

41. Through $(-2, 3, 2)$ and parallel to $3x + y + z = 4$.

42. Perpendicular to the vector $\vec{v} = 2\vec{i} - 3\vec{j} + 5\vec{k}$ and passing through the point $(4, 5, -2)$.

43. A plane has equation $z = 5x - 2y + 7$.

(a) Find a value of λ making the vector $\lambda\vec{i} + \vec{j} + 0.5\vec{k}$ normal to the plane.
(b) Find a value of a so that the point $(a + 1, a, a - 1)$ lies on the plane.

44. The points $(5, 0, 0)$, $(0, -3, 0)$, and $(0, 0, 2)$ form a triangle. Find the lengths of the sides of the triangle and each of its angles.

45. Let S be the triangle with vertices $A = (2, 2, 2)$, $B = (4, 2, 1)$, and $C = (2, 3, 1)$.

(a) Find the length of the shortest side of S.
(b) Find the cosine of the angle BAC at vertex A.

46. A basketball gymnasium is 25 meters high, 80 meters wide and 200 meters long. For a half time stunt, the cheerleaders want to run two strings, one from each of the two corners above one basket to the diagonally opposite corners of the gym floor. What is the cosine of the angle made by the strings as they cross?

47. A 100-meter dash is run on a track in the direction of the vector $\vec{v} = 2\vec{i} + 6\vec{j}$. The wind velocity $\vec{w}$ is $5\vec{i} + \vec{j}$ km/hr. The rules say that a legal wind speed measured in the direction of the dash must not exceed 5 km/hr. Will the race results be disqualified due to an illegal wind? Justify your answer.

62. Use the following steps and the results of Problems 59–60 to show (without trigonometry) that the geometric and algebraic definitions of the dot product are equivalent.

Let $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ be any vectors. Write $(\vec{u} \cdot \vec{v})_{\text{geom}}$ for the result of the dot product computed geometrically. Substitute $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and use Problems 59–60 to expand $(\vec{u} \cdot \vec{v})_{\text{geom}}$. Substitute for $\vec{v}$ and expand. Then calculate the dot products $\vec{i} \cdot \vec{i}, \vec{i} \cdot \vec{j}$, etc. geometrically.

63. For any vectors $\vec{v}$ and $\vec{w}$, consider the following function of t:

$$q(t) = (\vec{v} + t\vec{w}) \cdot (\vec{v} + t\vec{w}).$$

(a) Explain why $q(t) \geq 0$ for all real t.
(b) Expand $q(t)$ as a quadratic polynomial in t using the properties on page 702.
(c) Using the discriminant of the quadratic, show that,

$$|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\|\|\vec{w}\|.$$

13.4 THE CROSS PRODUCT

In the previous section we combined two vectors to get a number, the dot product. In this section we see another way of combining two vectors, this time to get a vector, the *cross product*. Any two vectors in 3-space form a parallelogram. We define the cross product using this parallelogram.

The Area of a Parallelogram

Consider the parallelogram formed by the vectors $\vec{v}$ and $\vec{w}$ with an angle of θ between them. Then Figure 13.35 shows

$$\text{Area of parallelogram} = \text{Base} \cdot \text{Height} = \|\vec{v}\|\|\vec{w}\| \sin\theta.$$

How would we compute the area of the parallelogram if we were given $\vec{v}$ and $\vec{w}$ in components, $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$? Project 1 on page 721 shows that if $\vec{v}$ and $\vec{w}$ are in the xy-plane, so $v_3 = w_3 = 0$, then

$$\text{Area of parallelogram} = |v_1w_2 - v_2w_1|.$$

What if $\vec{v}$ and $\vec{w}$ do not lie in the xy-plane? The cross product will enable us to compute the area of the parallelogram formed by any two vectors.

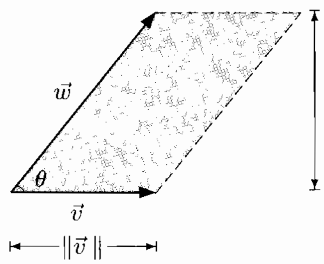


Figure 13.35: Parallelogram formed by $\vec{v}$ and $\vec{w}$ has Area $= \|\vec{v}\|\|\vec{w}\| \sin\theta$

Definition of the Cross Product

We define the cross product of the vectors $\vec{v}$ and $\vec{w}$, written $\vec{v} \times \vec{w}$, to be a vector perpendicular to both $\vec{v}$ and $\vec{w}$. The magnitude of this vector is the area of the parallelogram formed by the two vectors. The direction of $\vec{v} \times \vec{w}$ is given by the normal vector, $\vec{n}$, to the plane defined by $\vec{v}$ and $\vec{w}$. If we require that $\vec{n}$ be a unit vector, there are two choices for $\vec{n}$, pointing out of the plane in opposite directions. We pick one by the following rule (see Figure 13.36):

The right-hand rule: Place $\vec{v}$ and $\vec{w}$ so that their tails coincide and curl the fingers of your right hand through the smaller of the two angles from $\vec{v}$ to $\vec{w}$; your thumb points in the direction of the normal vector, $\vec{n}$.

Exercises and Problems for Section 13.4

Exercises

In Exercises 1–7, use the algebraic definition to find $\vec{v} \times \vec{w}$.

1. $\vec{v} = \vec{k}, \vec{w} = \vec{j}$
2. $\vec{v} = -\vec{i}, \vec{w} = \vec{j} + \vec{k}$
3. $\vec{v} = \vec{i} + \vec{k}, \vec{w} = \vec{i} + \vec{j}$
4. $\vec{v} = \vec{i} + \vec{j} + \vec{k}, \vec{w} = \vec{i} + \vec{j} + -\vec{k}$
5. $\vec{v} = 2\vec{i} - 3\vec{j} + \vec{k}, \vec{w} = \vec{i} + 2\vec{j} - \vec{k}$
6. $\vec{v} = 2\vec{i} - \vec{j} - \vec{k}, \vec{w} = -6\vec{i} + 3\vec{j} + 3\vec{k}$
7. $\vec{v} = -3\vec{i} + 5\vec{j} + 4\vec{k}, \vec{w} = \vec{i} - 3\vec{j} - \vec{k}$

Use the geometric definition in Exercises 8–9 to find:

8. $2\vec{i} \times (\vec{i} + \vec{j})$
9. $(\vec{i} + \vec{j}) \times (\vec{i} - \vec{j})$

In Exercises 10–11, use the properties on page 712 to find:

10. $((\vec{i} + \vec{j}) \times \vec{i}) \times \vec{j}$
11. $(\vec{i} + \vec{j}) \times (\vec{i} \times \vec{j})$

Find an equation for the plane through the points in Exercises 12–13.

12. $(1,0,0), (0,1,0), (0,0,1)$.
13. $(3,4,2), (-2,1,0), (0,2,1)$.
14. For $\vec{a} = 3\vec{i} + \vec{j} - \vec{k}$ and $\vec{b} = \vec{i} - 4\vec{j} + 2\vec{k}$, find $\vec{a} \times \vec{b}$ and check that it is perpendicular to both $\vec{a}$ and $\vec{b}$.
15. If $\vec{v} = 3\vec{i} - 2\vec{j} + 4\vec{k}$ and $\vec{w} = \vec{i} + 2\vec{j} - \vec{k}$, find $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$. What is the relation between the two answers?

Problems

16. Find a vector parallel to the line of intersection of the planes given by the equations $2x - 3y + 5z = 2$ and $4x + y - 3z = 7$.
17. Find the equation of the plane through the origin which is perpendicular to the line of intersection of the planes in Problem 16.
18. Find the equation of the plane through the point $(4, 5, 6)$ and perpendicular to the line of intersection of the planes in Problem 16.
19. Find an equation for the plane through the origin containing the points $(1, 3, 0)$ and $(2, 4, 1)$.
20. Find a vector parallel to the line of intersection of the two planes $4x - 3y + 2z = 12$ and $x + 5y - z = 25$.
21. Find a vector parallel to the intersection of the planes $2x - 3y + 5z = 2$ and $4x + y - 3z = 7$.
22. Find the equation of the plane through the origin which is perpendicular to the line of intersection of the planes in Problem 21.
23. Find the equation of the plane through the point $(4, 5, 6)$ which is perpendicular to the line of intersection of the planes in Problem 21.
24. Find the equation of a plane through the origin and perpendicular to $x - y + z = 5$ and $2x + y - 2z = 7$.
25. Let $P = (0,1,0), Q = (-1,1,2), R = (2,1,-1)$. Find
    (a) The area of the triangle PQR.
    (b) The equation for a plane that contains P, Q, and R.
26. Let $A = (-1,3,0)$, $B = (3,2,4)$, and $C = (1,-1,5)$.
    (a) Find an equation for the plane that passes through these three points.
    (b) Find the area of the triangle determined by these three points.
27. If $\vec{v}$ and $\vec{w}$ are both parallel to the xy-plane, what can you conclude about $\vec{v} \times \vec{w}$? Explain.
28. Suppose $\vec{v} \cdot \vec{w} = 5$ and $\|\vec{v} \times \vec{w}\| = 3$, and the angle between $\vec{v}$ and $\vec{w}$ is θ. Find
    (a) $\tan\theta$ (b) θ.
29. If $\vec{v} \times \vec{w} = 2\vec{i} - 3\vec{j} + 5\vec{k}$, and $\vec{v} \cdot \vec{w} = 3$, find $\tan\theta$ where θ is the angle between $\vec{v}$ and $\vec{w}$.
30. The point P in Figure 13.42 has position vector $\vec{v}$ obtained by rotating the position vector $\vec{r}$ of the point (x, y) by $90°$ counterclockwise about the origin.
    (a) Use the geometric definition of the cross product to explain why $\vec{v} = \vec{k} \times \vec{r}$.
    (b) Find the coordinates of P.

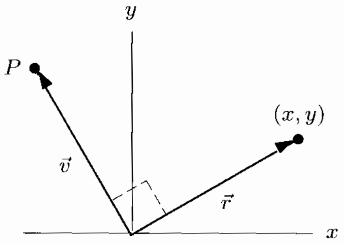


Figure 13.42

31. Use the algebraic definition to check that
$$\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}).$$
32. If $\vec{v}$ and $\vec{w}$ are nonzero vectors, use the geometric definition of the cross product to explain why
$$(\lambda\vec{v}) \times \vec{w} = \lambda(\vec{v} \times \vec{w}) = \vec{v} \times (\lambda\vec{w}).$$
Consider the cases $\lambda > 0$, and $\lambda = 0$, and $\lambda < 0$ separately.

33. Use a parallelepiped to show that $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$ for any vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$.

34. Show that $\|\vec{a}\times\vec{b}\|^2 = \|\vec{a}\|^2\|\vec{b}\|^2 - (\vec{a}\cdot\vec{b})^2$.

35. If $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, show that

$$\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}.$$

Geometrically, what does this imply about $\vec{a}$, $\vec{b}$, and $\vec{c}$?

36. If $\vec{a} = a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$, $\vec{b} = b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$ and $\vec{c} = c_1\vec{i}+c_2\vec{j}+c_3\vec{k}$ are any three vectors in space, show that

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

37. Use the fact that $\vec{i}\times\vec{i}=\vec{0}$, $\vec{i}\times\vec{j}=\vec{k}$, $\vec{i}\times\vec{k}=-\vec{j}$, and so on, together with the properties on page 712 to derive the algebraic definition for the cross product.

38. In this problem, we arrive at the algebraic definition for the cross product by a different route. Let $\vec{a} = a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ and $\vec{b} = b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$. We seek a vector $\vec{v} = x\vec{i}+y\vec{j}+z\vec{k}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$. Use this requirement to construct two equations for x, y, and z. Eliminate x and solve for y in terms of z. Then eliminate y and solve for x in terms of z. Since z can be any value whatsoever (the direction of $\vec{v}$ is unaffected), select the value for z which eliminates the denominator in the equation you obtained. How does the resulting expression for $\vec{v}$ compare to the formula we derived on page 711?

39. For vectors $\vec{a}$ and $\vec{b}$, let $\vec{c} = \vec{a}\times(\vec{b}\times\vec{a})$.

(a) Show that $\vec{c}$ lies in the plane containing $\vec{a}$ and $\vec{b}$.
(b) Use Problems 33 and 34 to show that $\vec{a}\cdot\vec{c}=0$ and $\vec{b}\cdot\vec{c} = \|\vec{a}\|^2\|\vec{b}\|^2 - (\vec{a}\cdot\vec{b})^2$.
(c) Show that

$$\vec{a}\times(\vec{b}\times\vec{a}) = \|\vec{a}\|^2\vec{b} - (\vec{a}\cdot\vec{b})\vec{a}.$$

40. Use the result of Problem 33 to show that the cross product distributes over addition. First, use distributivity for the dot product to show that for any vector $\vec{d}$,

$$[(\vec{a}+\vec{b})\times\vec{c}]\cdot\vec{d} = [(\vec{a}\times\vec{c})+(\vec{b}\times\vec{c})]\cdot\vec{d}.$$

Next, show that for any vector $\vec{d}$,

$$[((\vec{a}+\vec{b})\times\vec{c})-(\vec{a}\times\vec{c})-(\vec{b}\times\vec{c})]\cdot\vec{d}=0.$$

Finally, explain why you can conclude that

$$(\vec{a}+\vec{b})\times\vec{c} = (\vec{a}\times\vec{c})+(\vec{b}\times\vec{c}).$$

41. Figure 13.43 shows the tetrahedron determined by three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$. The *area vector* of a face is a vector perpendicular to the face, pointing outward, whose magnitude is the area of the face. Show that the sum of the four outward pointing area vectors of the faces equals the zero vector.

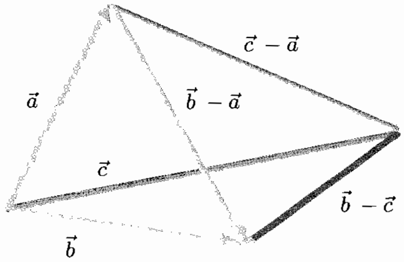


Figure 13.43

In Problems 42–44, find the vector representing the area of a surface. The magnitude of the vector equals the magnitude of the area; the direction is perpendicular to the surface. Since there are two perpendicular directions, we pick one by giving an orientation for the surface.

42. The rectangle with vertices $(0,0,0)$, $(0,1,0)$, $(2,1,0)$, and $(2,0,0)$, oriented so that it faces downward.

43. The circle of radius 2 in the yz-plane, facing in the direction of the positive x-axis.

44. The triangle ABC, oriented upward, where $A=(1,2,3)$, $B=(3,1,2)$, and $C=(2,1,3)$.

45. This problem relates the area of a parallelogram S lying in the plane $z=mx+ny+c$ to the area of its projection R in the xy-plane. Let S be determined by the vectors $\vec{u} = u_1\vec{i}+u_2\vec{j}+u_3\vec{k}$ and $\vec{v} = v_1\vec{i}+v_2\vec{j}+v_3\vec{k}$. See Figure 13.44.

(a) Find the area of S
(b) Find the area of R
(c) Find m and n in terms of the components of $\vec{u}$ and $\vec{v}$.
(d) Show that

$$\text{Area of } S = \sqrt{1+m^2+n^2}\cdot\text{Area of } R$$

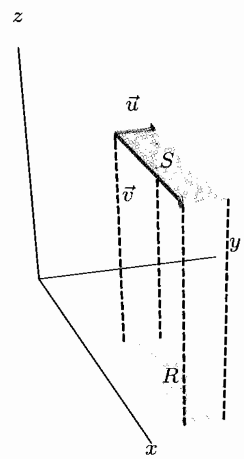


Figure 13.44

CHAPTER SUMMARY (see also Ready Reference at the end of the book)

- **Vectors**
  Geometric definition of vector addition, subtraction and scalar multiplication, resolving into $\vec{i}$, $\vec{j}$, and $\vec{k}$ components, magnitude of a vector, algebraic properties of addition and scalar multiplication.
- **Dot Product**
  Geometric and algebraic definition, algebraic properties, using dot products to find angles and determine perpendicularity, the equation of a plane with given normal vector passing through a given point, projection of a vector in a direction given by a unit vector.
- **Cross Product**
  Geometric and algebraic definition, algebraic properties, cross product and volume, finding the equation of a plane through three points.

REVIEW EXERCISES AND PROBLEMS FOR CHAPTER THIRTEEN

Exercises

In Exercises 1–2, is the quantity a vector or a scalar? Compute it.

1. $\vec{u} \cdot \vec{v}$, where $\vec{u} = 2\vec{i} - 3\vec{j} - 4\vec{k}$ and $\vec{v} = \vec{k} - \vec{j}$

2. $\vec{u} \times \vec{v}$, where $\vec{u} = 2\vec{i} - 3\vec{j} - 4\vec{k}$ and $\vec{v} = 3\vec{i} - \vec{j} + \vec{k}$.

3. Resolve the vectors in Figure 13.45 into components.

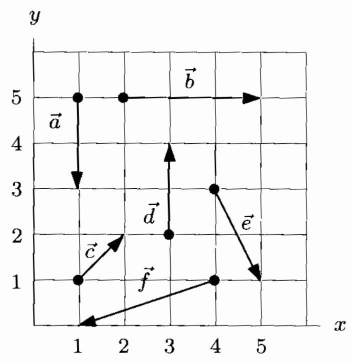


Figure 13.45

4. Resolve vector $\vec{v}$ into components if $\|\vec{v}\| = 8$ and the direction of $\vec{v}$ is shown in Figure 13.46.

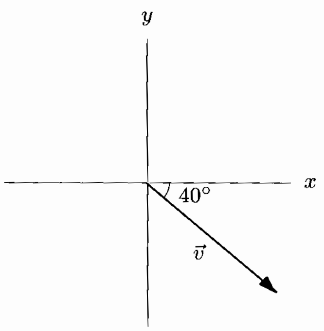


Figure 13.46

For Exercises 5–7, perform the indicated operations on the following vectors:

$$\vec{c} = \vec{i} + 6\vec{j}, \quad \vec{x} = -2\vec{i} + 9\vec{j}, \quad \vec{y} = 4\vec{i} - 7\vec{j}.$$

5. $5\vec{c}$ **6.** $\vec{c} + \vec{x} + \vec{y}$ **7.** $\|\vec{x} - \vec{c}\|$

In Exercises 8–17, use $\vec{v} = 2\vec{i} + 3\vec{j} - \vec{k}$ and $\vec{w} = \vec{i} - \vec{j} + 2\vec{k}$ to calculate the given quantities.

8. $\vec{v} + 2\vec{w}$

9. $3\vec{v} - \vec{w} - \vec{v}$

10. $\|\vec{v} + \vec{w}\|$

11. $\vec{v} \cdot \vec{w}$

12. $\vec{v} \times \vec{w}$

13. $\vec{v} \times \vec{v}$

14. $(\vec{v} \cdot \vec{w})\vec{v}$

15. $(\vec{v} \times \vec{w}) \cdot \vec{w}$

16. $(\vec{v} \times \vec{w}) \times \vec{w}$

17. $(\vec{v} \times \vec{w}) \times (\vec{v} \times \vec{w})$

In Exercises 18–19, find a normal vector to the plane.

18. $2x + y - z = 5$

19. $2(x - z) = 3(x + y)$

20. Find the equation of the plane through the origin which is parallel to $z = 4x - 3y + 8$.

21. Let $\vec{v} = 3\vec{i} + 2\vec{j} - 2\vec{k}$ and $\vec{w} = 4\vec{i} - 3\vec{j} + \vec{k}$. Find each of the following:

(a) $\vec{v} \cdot \vec{w}$
(b) $\vec{v} \times \vec{w}$
(c) A vector of length 5 parallel to vector $\vec{v}$
(d) The angle between vectors $\vec{v}$ and $\vec{w}$
(e) The component of $\vec{v}$ in the direction of $\vec{w}$
(f) A vector perpendicular to vector $\vec{v}$
(g) A vector perpendicular to both vectors $\vec{v}$ and $\vec{w}$

In Exercises 22–27, find a vector with the given property.

22. Length 10, parallel to $2\vec{i} + 3\vec{j} - \vec{k}$.

23. Unit vector perpendicular to $\vec{i} + \vec{j}$ and $\vec{i} - \vec{j} - \vec{k}$

24. Unit vector in the xy-plane perpendicular to $3\vec{i} - 2\vec{j}$.

25. A vector normal to $4(x-1) + 6(z+3) = 12$.

26. The vector obtained from $4\vec{i} + 3\vec{j}$ by rotating it 90° counterclockwise.

27. A nonzero vector perpendicular to $\vec{v} = 3\vec{i} - \vec{j} + \vec{k}$ and $\vec{w} = \vec{i} - 2\vec{j} + \vec{k}$.

28. Which of the following vectors are parallel?

$$\vec{u} = 2\vec{i} + 4\vec{j} - 2\vec{k}, \quad \vec{p} = \vec{i} + \vec{j} + \vec{k},$$
$$\vec{v} = \vec{i} - \vec{j} + 3\vec{k}, \quad \vec{q} = 4\vec{i} - 4\vec{j} + 12\vec{k},$$
$$\vec{w} = -\vec{i} - 2\vec{j} + \vec{k}, \quad \vec{r} = \vec{i} - \vec{j} + \vec{k}.$$

In Exercises 29–34, find the parallel and perpendicular components of the force vector $\vec{F}$ in the direction of the displacement vector $\vec{d}$. Then find the work W done by $\vec{F}$ though the displacement $\vec{d}$.

29. $\vec{F} = 2\vec{i} + 4\vec{j}, \quad \vec{d} = \vec{i} + 2\vec{j}$

30. $\vec{F} = -2\vec{i} - 4\vec{j}, \quad \vec{d} = \vec{i} + 2\vec{j}$

31. $\vec{F} = 2\vec{i} + 4\vec{j}, \quad \vec{d} = 2\vec{i} - 1\vec{j}$

32. $\vec{F} = 2\vec{i} + 4\vec{j}, \quad \vec{d} = 3\vec{i} - 4\vec{j}$

33. $\vec{F} = 2\vec{i}, \quad \vec{d} = \vec{i} + \vec{j}$

34. $\vec{F} = 5\vec{i} + 2\vec{j}, \quad \vec{d} = 3\vec{j}$

Problems

35. Figure 13.47 shows a rectangular box containing several vectors. Are the following statements true or false? Explain.

(a) $\vec{c} = \vec{f}$ **(b)** $\vec{a} = \vec{d}$ **(c)** $\vec{a} = -\vec{b}$
(d) $\vec{g} = \vec{f} + \vec{a}$ **(e)** $\vec{e} = \vec{a} - \vec{b}$ **(f)** $\vec{d} = \vec{g} - \vec{c}$

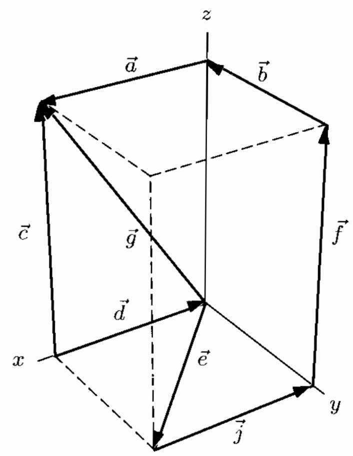


Figure 13.47

36. Shortly after takeoff, a plane is climbing northwest through still air at an airspeed of 200 km/hr, and rising at a rate of 300 m/min. Resolve its velocity vector into components. The x-axis points east, the y-axis points north, and the z-axis points up.

37. A boat is heading due east at 25 km/hr (relative to the water). The current is moving toward the southwest at 10 km/hr.

(a) Give the vector representing the actual movement of the boat.
(b) How fast is the boat going, relative to the ground?
(c) By what angle does the current push the boat off of its due east course?

38. A model rocket is shot into the air at an angle with the earth of about 60°. The rocket is going fast initially but slows down as it reaches its highest point. It picks up speed again as it falls to earth.

(a) Sketch a graph showing the path of the rocket. Draw several velocity vectors on your graph.
(b) A second rocket has a parachute that deploys as it begins its descent. How do the velocity vectors from part (a) change for this rocket?

39. A car drives clockwise around the track in Figure 13.48, slowing down at the curves and speeding up along the straight portions. Sketch velocity vectors at the points P, Q, and R.

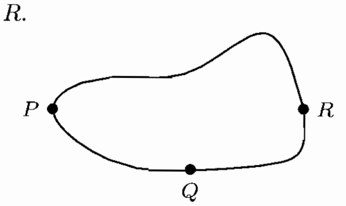


Figure 13.48

40. A racing car drives clockwise around the track shown in Figure 13.48 at a constant speed. At what point on the track does the car have the longest acceleration vector, and in roughly what direction is it pointing? (Recall that acceleration is the rate of change of velocity.)

41. List any vectors which are parallel to each other and any vectors which are perpendicular to each other:

$\vec{v}_1 = \vec{i} - 2\vec{j}$ $\quad \vec{v}_2 = 2\vec{i} + 4\vec{j}$
$\vec{v}_3 = 3\vec{i} + 1.5\vec{j}$ $\quad \vec{v}_4 = -1.2\vec{i} + 2.4\vec{j}$
$\vec{v}_5 = -5\vec{i} - 2.5\vec{j}$ $\quad \vec{v}_6 = 12\vec{i} - 12\vec{j}$
$\vec{v}_7 = 4\vec{i} + 2\vec{j}$ $\quad \vec{v}_8 = 3\vec{i} - 6\vec{j}$
$\vec{v}_9 = 0.70\vec{i} - 0.35\vec{j}$

42. An object is to be moved vertically upward by a crane. As the crane cannot get directly above the object, three ropes are attached to guide the object. One rope is pulled parallel to the ground with a force of 100 newtons in a direction 30° north of east. The second rope is pulled parallel to the ground with a force of 70 newtons in a direction 80° south of east. If the crane is attached to the third rope and can pull with a total force of 3000 newtons, find the force vector for the crane. What is the resulting (total) force on the object? (Assume vector $\vec{i}$ points east, vector $\vec{j}$ points north, and vector $\vec{k}$ points vertically up.)

43. For what values of t are $\vec{u} = t\vec{i} - \vec{j} + \vec{k}$ and $\vec{v} = t\vec{i} + t\vec{j} - 2\vec{k}$ perpendicular? Are there values of t for which $\vec{u}$ and $\vec{v}$ are parallel?

In Problems 44–45, find an equation of a plane that satisfies the given conditions.

44. Perpendicular to the vector $-\vec{i} + 2\vec{j} + \vec{k}$ and passing through the point $(1, 0, 2)$.

45. Perpendicular to the vector $5\vec{i} + \vec{j} - 2\vec{k}$ and passing through the point $(0, 1, -1)$.

46. Let $A = (0, 4)$, $B = (-1, -3)$, and $C = (-5, 1)$. Draw triangle ABC and find each of its interior angles.

47. Find the area of the triangle with vertices $P = (-2, 2, 0)$, $Q = (1, 3, -1)$, and $R = (-4, 2, 1)$.

48. A plane is drawn through the points $A = (2, 1, 0)$, $B = (0, 1, 3)$ and $C = (1, 0, 1)$. Find

(a) Two vectors lying in the plane.
(b) A vector perpendicular to the plane.
(c) The equation of the plane.

49. Given the points $P = (1, 2, 3)$, $Q = (3, 5, 7)$, and $R = (2, 5, 3)$, find:

(a) A unit vector perpendicular to a plane containing P, Q, R.
(b) The angle between PQ and PR.
(c) The area of the triangle PQR.
(d) The distance from R to the line through P and Q.

50. Find the distance from the point $P = (2, -1, 3)$ to the plane $2x + 4y - z = -1$.

51. Find an equation of the plane passing through the three points $(1, 1, 1)$, $(1, 4, 5)$, $(-3, -2, 0)$. Find the distance from the origin to the plane.

52. Suppose $\vec{v} \cdot \vec{w} = 8$ and $\vec{v} \times \vec{w} = 12\vec{i} - 3\vec{j} + 4\vec{k}$ and that the angle between $\vec{v}$ and $\vec{w}$ is θ. Find

(a) $\tan\theta$ **(b)** θ

53. Consider the plane $5x - y + 7z = 21$.

(a) Find a point on the x-axis on this plane.
(b) Find two other points on the plane.
(c) Find a vector perpendicular to the plane.
(d) Find a vector parallel to the plane.

54. An airport is at the point $(200, 10, 0)$ and an approaching plane is at the point $(550, 60, 4)$. Assume that the xy-plane is horizontal, with the x-axis pointing eastward and the y-axis pointing northward. Also assume that the z-axis is upward and that all distances are measured in kilometers. The plane flies due west at a constant altitude at a speed of 500 km/hr for half an hour. It then descends at 200 km/hr, heading straight for the airport.

(a) Find the velocity vector of the plane while it is flying at constant altitude.
(b) Find the coordinates of the point at which the plane starts to descend.
(c) Find a vector representing the velocity of the plane when it is descending.

55. Find the vector $\vec{v}$ with all of the following properties:

- Magnitude 10
- Angle of 45° with positive x-axis
- Angle of 75° with positive y-axis
- Positive $\vec{k}$ -component.

56. Two lines in space are skew if they are not parallel and do not intersect. Determine the minimum distance between two such lines.

57. (a) A vector $\vec{v}$ of magnitude v makes an angle α with the positive x-axis, angle β with the positive y-axis, and angle γ with the positive z-axis. Show that

$$\vec{v} = v\cos\alpha\vec{i} + v\cos\beta\vec{j} + v\cos\gamma\vec{k}.$$

(b) $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called *direction cosines*. Show that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

58. Three people are trying to hold a ferocious lion still for the veterinarian. The lion, in the center, is wearing a collar with three ropes attached to it and each person has hold of a rope. Charlie is pulling in the direction 62° west of north with a force of 175 newtons and Sam is pulling in the direction 43° east of north with a force of 200 newtons. What is the direction and magnitude of the force which must be exerted by Alice on the third rope to counterbalance Sam and Charlie?

CAS Challenge Problems

59. Let $\vec{a} = x\vec{i} + y\vec{j} + z\vec{k}$, $\vec{b} = u\vec{i} + v\vec{j} + w\vec{k}$, and $\vec{c} = m\vec{a} + n\vec{b}$. Compute $(\vec{a} \times \vec{b}) \cdot \vec{c}$ and $(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c})$, and explain the geometric meaning of your answers.

60. Let $\vec{a} = x\vec{i} + y\vec{j} + z\vec{k}$, $\vec{b} = u\vec{i} + v\vec{j} + w\vec{k}$ and $\vec{c} = r\vec{i} + s\vec{j} + t\vec{k}$. Show that the parallelepiped with edges $\vec{a}$, $\vec{b}$, $\vec{c}$ has the same volume as the parallelepiped with edges $\vec{a}$, $\vec{b}$, $2\vec{a} - \vec{b} + \vec{c}$. Explain this result geometrically.