MATH 224 Examination 2 May 10, 2010

Name Key

GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS.
There are four problems. Each is worth 25 points.

1. A contour diagram of a continuous function $f(x, y)$ with continuous partial derivatives is given in Figure 1.
   (a) Determine the signs of the following partial derivatives
   $$f_x(P),\ f_y(P),\ f_{xx}(P),\ f_{yy}(P),\ f_{xy}(P),\ f_{yx}(P).$$
   (b) Determine the signs of the following partial derivatives
   $$f_x(Q),\ f_y(Q),\ f_{xx}(Q),\ f_{yy}(Q),\ f_{xy}(Q),\ f_{yx}(Q).$$
   (c) Sketch the vectors $(\nabla f)(P)$ and $(\nabla f)(Q)$ in Figure 1.
   (d) Which of the vectors $(\nabla f)(P)$, $(\nabla f)(Q)$ has larger magnitude? Explain.

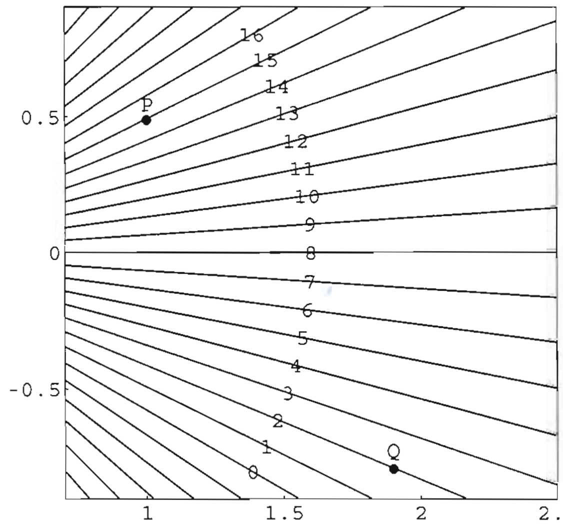


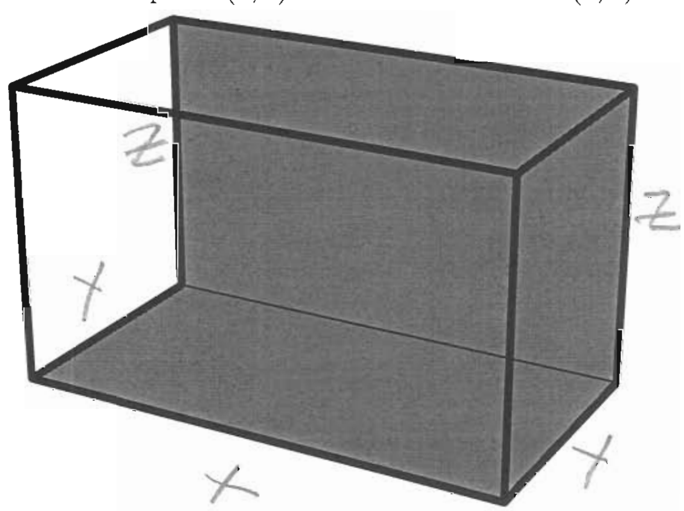
Figure 1: A contour diagram of $f(x, y)$

2. Consider the function $g(x, y) = 4y\sqrt{x} - 5\ln y$.
   (a) Find the average rate of change of g as you go from $(4, 1)$ to $(1, 5)$ along the line segment joining these two points.
   (b) Find the instantaneous rate of change of g as you leave the point $(4, 1)$ heading toward $(1, 5)$.
   (c) Find the instantaneous rate of change of g as you arrive at the point $(1, 5)$ from the direction of $(4, 1)$.

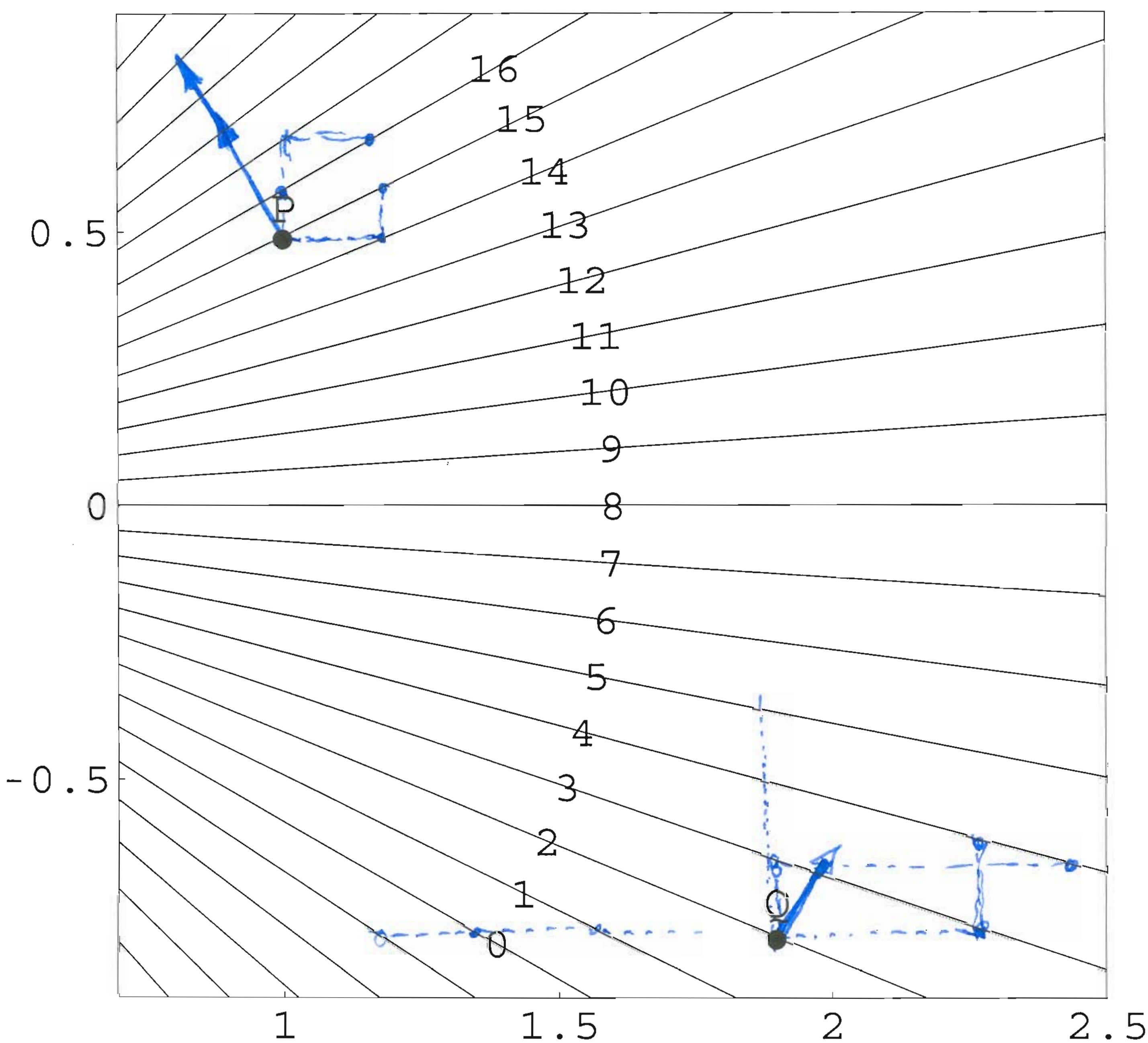
3. The goal of this problem is to make the cheapest display box with the volume of 1 cubic meter. As you can see in the figure to the right, **four** sides of the display box are made of high quality glass, while the bottom of the box and one vertical side are made of metal. If glass costs four times as much (per square meter) as metal, find the dimensions of the display box that will minimize the cost of the materials.

   Give both: exact and approximate values for the dimensions of the box.

   Use the second derivative test to confirm that the point you obtained is a local minimum.



4. Captain Astro is in trouble near the sunny side of Mercury. She is at location $(1, 1, 1)$, and the temperature of the ship's hull when she is at location (x, y, z) will be given by $T(x, y, z) = e^{-x^2-2y^2-2z^2}$, where x, y, and z are measured in meters and the temperature is measured in degrees (these are some special very hot degrees).
   (a) In what direction should she proceed in order to decrease the temperature most rapidly?
   (b) If the ship travels at e^5 meters per second, how fast (in degrees per second) will the temperature decrease if she proceeds in that direction?
   (c) Assume again that the ship travels at e^5 meters per second. Calculate how fast will temperature change if Captain Astro decides to proceed in the direction $\vec{i} + \vec{j} + \vec{k}$. Pay attention to the sign of the change.
   (d) Give one direction in which the rate of change of temperature will be 0.


16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
P
Q
0.5
0
-0.5
1
1.5
2
2.5

$\boxed{1}$

(1) (a) $f_x(P) < 0$ goes from -1 to $-1/2$

$f_y(P) > 0$

$f_{xx}(P) > 0$ $\quad f_{yy}(P) < 0$

$f_{xy}(P) < 0$ $\quad f_{yx}(P) < 0$

(b) $f_x(Q) > 0$ $\quad f_y(Q) > 0$

$f_{xx}(Q) < 0$ $\quad f_{yy}(Q) > 0$

$f_{xy}(Q) < 0$ $\quad f_{yx}(Q) < 0$

$+1$ to $+\frac{1}{2}$

(c) see fig.

(d) $\vec{\nabla} f(P)$ is much longer since the function ~~since~~ run ~~for~~ for rise 1 is much smaller.

② ⓐ $\cancel{f_x}$ $\boxed{2}$

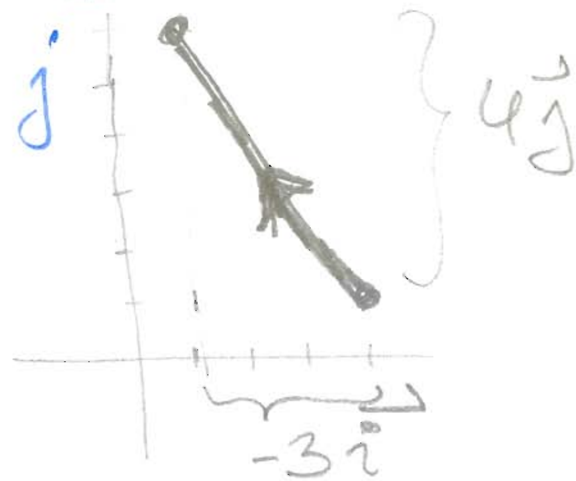
the run from $(4,1)$ to $(1,5)$ is 5.

rise is $4\cdot 5\cdot\sqrt{1} - 5\ln 5 - (4\cdot 1\cdot 2 - 0)$

$= 12 - 5\ln 5$

$\frac{\text{rise}}{\text{run}} = \boxed{\frac{12}{5} - \ln 5}$ is the average rate of change

ⓑ $\vec{u} = -\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$



$f_x = \frac{2y}{\sqrt{x}}$ $\quad \frac{2\cdot 1}{2} = 1$

$f_y = 4\sqrt{x} - \frac{5}{y}$ $\quad 4\cdot 2 - \frac{5}{1}$

$(\nabla f)(4,1) = \vec{i} + 3\vec{j}$

$(\nabla f)(4,1)\cdot\vec{u} = -\frac{3}{5} + \frac{12}{5} = \frac{9}{5}$

ⓒ $(\nabla f)(1,5) = 10\vec{i} + 3\vec{j}$

$(\nabla f)(1,5)\cdot\vec{u} = -\frac{30}{5} + \frac{12}{5} = -\frac{18}{5}$

3 COST: metal $x(y+z)$ $\boxed{3}$

glass $4(2yz + x(y+z))$

Total cost

$$\cancel{f(x,y)} = xy + xz + 8yz + 4xy + 4xz$$

$$= 5xy + 5xz + 8yz$$

Cost $= \boxed{5xy + (5x+8y)z}$

$\boxed{xyz = 1}$ $\boxed{z = \frac{1}{xy}}$

$$f(x,y) = 5xy + (5x+8y)\frac{1}{xy}$$

$$= \boxed{5xy + \frac{5}{y} + \frac{8}{x}}$$

$$f_x = 5y - \frac{8}{x^2}$$

$$f_y = 5x - \frac{5}{y^2} \longrightarrow \boxed{x = \frac{1}{y^2}}$$

$$\frac{1}{x^2} = y^4$$

$$\boxed{5y - 8y^4 = 0}$$

$$y^3 = \frac{5}{8} \quad \boxed{y = \frac{\sqrt[3]{5}}{2}}$$

3

$$x = \frac{1}{y^2} = \frac{2^2}{\sqrt[3]{5^2}} = \frac{4}{\sqrt[3]{25}} \quad \boxed{4}$$

$$z = \frac{1}{\frac{\cancel{4}^{2}}{\sqrt[3]{\cancel{25}_{5}}} \cdot \frac{\cancel{\sqrt[3]{5}}^{1}}{\cancel{2}_{1}}} = \frac{\sqrt[3]{5}}{2}$$

$$z = y = \boxed{\frac{\sqrt[3]{5}}{2}}, \quad x = \boxed{\frac{4}{\sqrt[3]{25}}}$$

$\boxed{0.85499}$ $\boxed{1.36798}$

$$f_{xx} = + \frac{16}{x^3} > 0 \quad f_{xy} = 5 \quad \boxed{y = \frac{5}{8}x}$$

$$f_{yy} = \frac{10}{y^3}$$

$$D = \frac{16}{\frac{2^3\, 4^3}{25}} \cdot \frac{10}{\frac{5\,1}{2^3}} - 25 =$$

$$= \frac{\cancel{160}^{20} * 5}{81} - 25 = \cancel{2}75 > 0$$

$D > 0 \quad f_{xx} > 0 \Rightarrow \boxed{\text{loc. min}}$

(4) (a)

5

$T_x = e^{-x^2-2y^2-2z^2} \cdot (-2x)$

$T_x(1,1,1) = e^{-5}(-2)$

$T_y(1,1,1) = e^{-5}(-4)$

$T_z(1,1,1) = e^{-5}(-4)$

$(\vec{\nabla} T)(1,1,1) = \boxed{-2e^{-5}(\vec{i}+2\vec{j}+2\vec{k})}$

to decrease go in direction

$\vec{i}+2\vec{j}+2\vec{k}$

(b) The temp will decrease at the rate

$-e^{5} \cdot \|\vec{\nabla} T(1,1,1)\| = -2\sqrt{1+4+4} = -6$

-6 °/meter.

ⓒ

$$e^5\left(\vec{\nabla}T \cdot \overbrace{\frac{1}{\sqrt{3}}(\vec{i}+\vec{j}+\vec{k})}^{\|\vec{u}\|=1}\right)$$

$\boxed{6}$

$$= -\frac{2}{\sqrt{3}}(1+2+2) = \frac{-10}{\sqrt{3}}$$

ⓓ $-4\vec{i}+\vec{j}+\vec{k}$

for example.

any $a\vec{i}+b\vec{j}+c\vec{k}$ such that

$a+2b+2c=0.$