MATH 224 Examination 3 May 28, 2010 Name Key

GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS.
There are five problems. Each is worth 20 points.

1. Consider the function $f(x, y, z) = x + y + z^2$.

   (a) Use Lagrange multipliers to find the maximum and minimum value of f subject to the constraint $x^2 + y^2 + z^2 = 1$.

   (b) Provide a geometric illustration in xy-plane of the significance of at least two points found in (1a).

2. Figure 1 below is a contour diagram of the function $f(x, y) = 1 - x^2 + y^2$. Four out of the following six regions in the xy-plane are represented in Figures 2 through 5.

$$D_1 = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le 1\} \qquad D_2 = \{(x, y) : 0 \le x,\ 0 \le y,\ x - y \le 1\}$$
$$D_3 = \{(x, y) : 0 \le x,\ 0 \le y,\ x + y \le 1\} \qquad D_4 = \{(x, y) : 0 \le x \le y \le 1\}$$
$$D_5 = \{(x, y) : 0 \le y \le x \le 1\} \qquad D_6 = \{(x, y) : x \le 1,\ y \le 1,\ x + y \ge 1\}$$

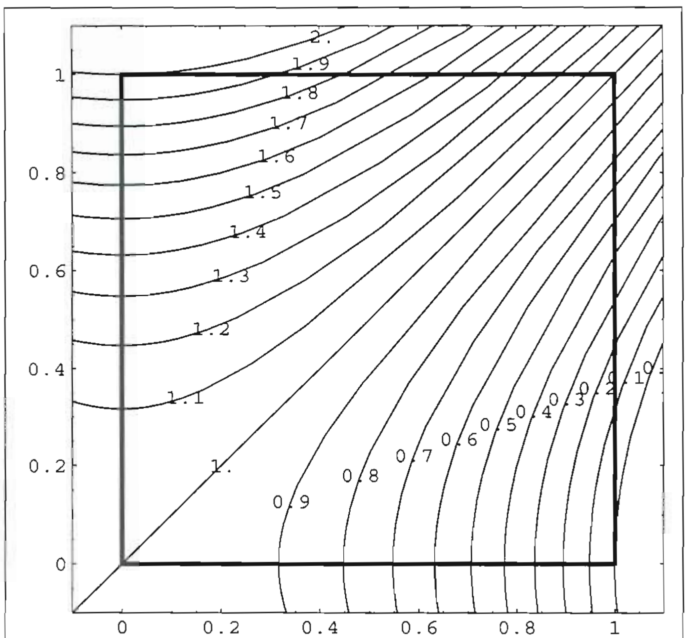


Figure 1: The function $f(x, y)$

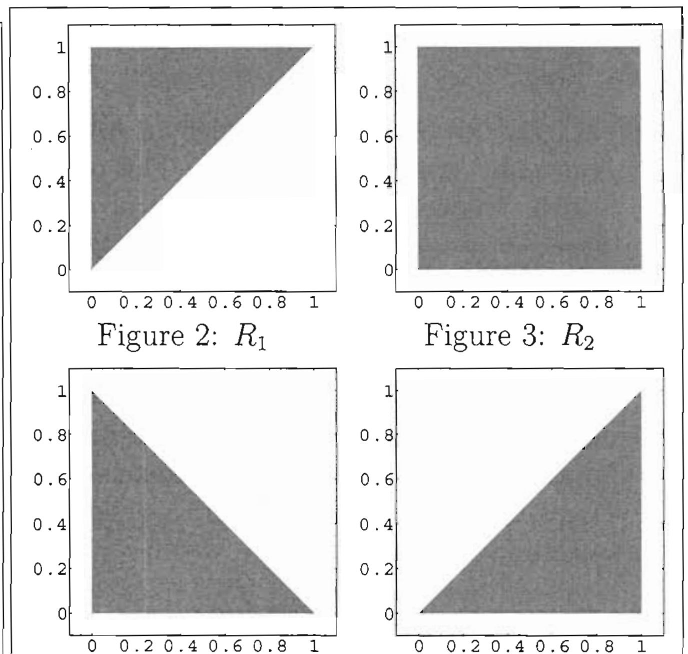


Figure 2: R_1 Figure 3: R_2 Figure 4: R_3 Figure 5: R_4

   (a) Match four of the regions $D_1, D_2, D_3, D_4, D_5, D_6$ with the regions R_1, R_2, R_3, and R_4 given in Figures 2 through 5.

   (b) Consider the integrals $I_1 = \int_{R_1} f\,dA$, $I_2 = \int_{R_2} f\,dA$, $I_3 = \int_{R_3} f\,dA$, $I_4 = \int_{R_4} f\,dA$. Order the numbers I_1, I_2, I_3, I_4, from the smallest to the largest.

   (c) Calculate $\int_{R_1} f\,dA$.

   (d) Calculate one of the remaining three integrals. (All three will earn you extra credit.)

OVER

3. A house is built on the xy-plane. The foundation of this house is the unit disk. The formula for the roof is given by $z = e^{-(x^2+y^2)}$.

   (a) Calculate the volume of this house. Give the exact value.

   (b) Give a simple geometric explanation why your answer must be greater than π/e and smaller than π.

   For your convenience the foundation is pictured in Figure 6.

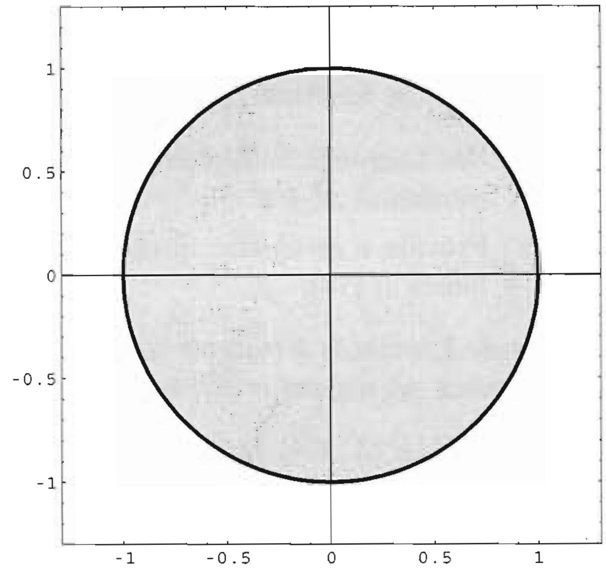


Figure 6

4. In this problem we consider a spherical chocolate truffle of radius 1. The outside of this chocolate truffle is made of chocolate, so it is quite dense. In fact the outside has density 1. The inside is filled with cream of varying density. Closer to the center cream gets thinner and thinner. In fact, the density of the cream at a point P inside the chocolate truffle equals to the distance of the point P to the center of the chocolate truffle. Calculate the mass of this chocolate truffle.

   Figure 7 gives a possible cross section of this chocolate truffle.

Figure 7

1 (a)

$\boxed{1}$

$1 = \lambda 2x$

$1 = \lambda 2y$

$2z = \lambda 2z$ $\quad \boxed{z=0}$, or $\boxed{\lambda = 1}$

$x^2+y^2+z^2 = 1$

Case $z = 0$

$x = \frac{1}{2\lambda}$, $y = \frac{1}{2\lambda}$ $\quad \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 1$

$\left(\frac{1}{2\lambda}\right)^2 = \frac{1}{2}$

$\frac{1}{4\lambda^2} = \frac{1}{2}$

$\lambda^2 = 1/2$

$\lambda = \pm \frac{1}{\sqrt{2}}$

~~#~~ Points:

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ P_1

$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ P_2

Case $\lambda = 1$ $\quad x = 1/2$, $y = 1/2$

$\frac{1}{4} + \frac{1}{4} + z^2 = 1$, $z^2 = \frac{1}{2}$

$z = \pm \frac{1}{\sqrt{2}}$

Points

$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ P_3

$\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}}\right)$ P_4

The values of f $\boxed{2}$

$$f(P_1) = \sqrt{2}$$

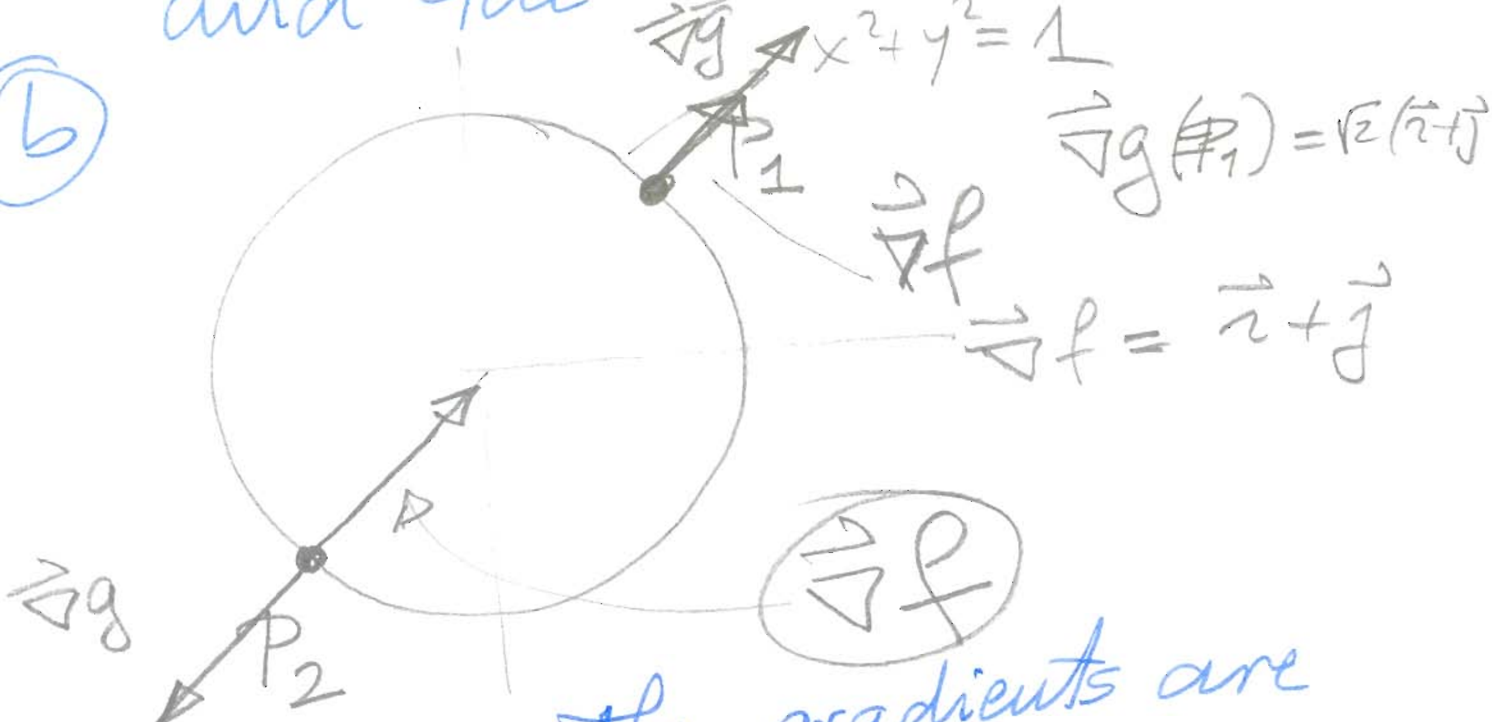
$$f(P_2) = -\sqrt{2}$$

$$f(P_3) = \frac{3}{2}$$

$$f(P_4) = \frac{3}{2}$$

So the maximum is $3/2$ and the minimum is $\sqrt{2}$.

(b)



the gradients are colinear!

$\boxed{3}$

2 (a) $D_1 = R_2$

$D_3 = R_3$

$D_4 = R_1$

$D_5 = R_4$

(b) $I_4 \leq I_3 \leq I_1 \leq I_2$

(c) $$I_1 = \int_0^1 \left(\int_x^1 (1 - x^2 + y^2)\, dy \right) dx$$

$$= \int_0^1 \left[(\cancel{1} - \cancel{x}) - x^2(1-x) + \frac{1}{3}(1 - x^3) \right] dx$$

$$= \int_0^1 \left(1 - x - x^2 + x^3 + \frac{1}{3} - \frac{1}{3}x^3 \right) dx =$$

$$\frac{2}{3}$$

$$= \frac{4}{3} - \frac{1}{2} - \frac{1}{3} + \frac{\cancel{2}\,1}{3} \cdot \frac{1}{\cancel{4}\,2} = \frac{4}{3} - \frac{3 + 2 \cdot 1}{6} = \frac{2}{3}$$

(d) $I_2 = 1$ $\boxed{4}$

$I_3 = 1/2$

$I_4 = 1 - 2/3 = 1/3$

$\frac{1}{3} < 1/2 < 2/3 < 1$

$I_4 \quad I_3 \quad I_1 \quad I_2$

(3) (a) $\iint e^{-(x^2+y^2)} dA = \int_0^{2\pi}\int_0^{1} e^{-r^2} r\,dr\,d\theta$

$A = \int_0^{2\pi}\left(\int_0^{1} \frac{1}{2} e^{-u} du\right) d\theta = \frac{1}{2} 2\pi \left(-e^{-u}\right)\Big|_0^1$

$= \pi\left(1 - e^{-1}\right)$

$r^2 = u$

$2r\,dr = du$

$= \pi - \frac{\pi}{e}$

$\pi - \frac{\pi}{e} > \frac{\pi}{e}$

$\pi > \frac{2\pi}{e}$ $\quad 1 > \frac{2}{e}$ $\quad e > 2$

(b) The ~~r~~lowest level of the roof of this house is $e^{-1} = \frac{1}{e}$ $\boxed{5}$

Thus the volume is $> \frac{\pi}{e}$.

The highest point on the roof is 1 so the volume is $< \pi$.

(4) $$\iiint_W \sqrt{x^2+y^2+z^2}\, dV =$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= 2\pi * 2 * \frac{1}{4} = \pi$$ compare to $\frac{4}{3}\pi$

The mass must be ≤ volume ~~smaller, but~~ One would expect even ~~more~~ smaller than π.