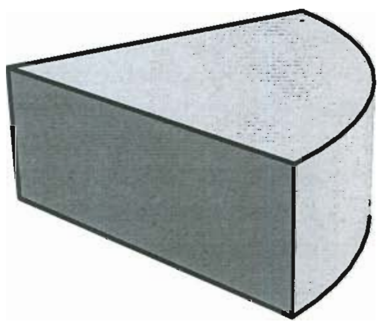
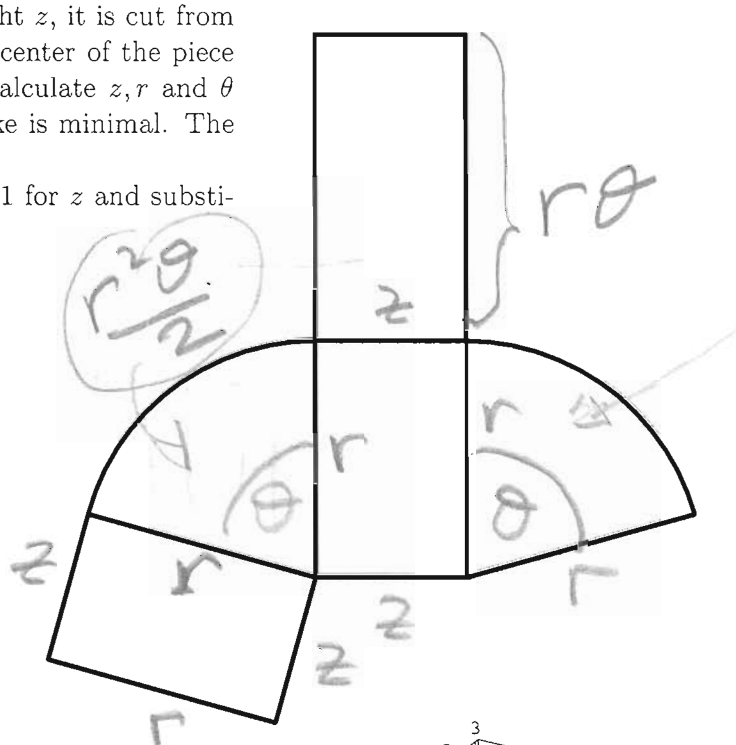
MATH 224 Final Examination
June 8, 2010

Name Key

GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS. There are eight problems. Each is worth 12.5 points.

1. A piece of cake is shown below. This cake has height z, it is cut from a cylindrical cake of radius r and the angle at the center of the piece is θ. Assume that the volume of the cake is 1. Calculate z, r and θ for which the corresponding surface area of the cake is minimal. The surface area of the cake is pictured to the right.
   Hint: Do not use Lagrange multipliers. Solve $V = 1$ for z and substitute z in the formula for the surface area.





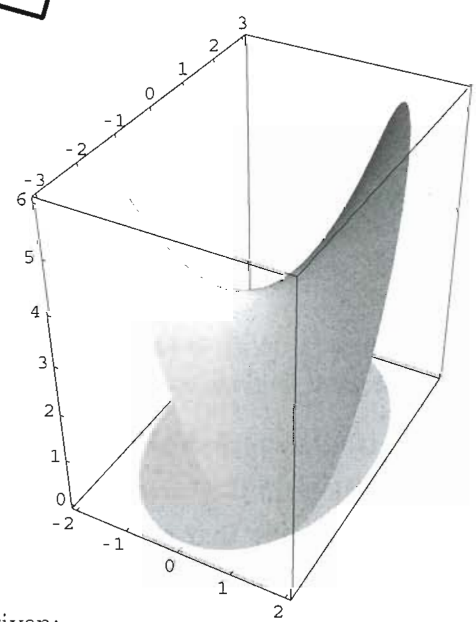
2. A house is built on the foundation bounded by the ellipse

$$9x^2 + 4y^2 = 36.$$

   The roof of the house is given by the equation $f(x, y) = 3 + xy$. Determine the volume of this house. The change of variables

$$x = 2s, \quad y = 3t$$

   can be very useful here.
   The roof and the foundation of this house are shown in the figure to the right.



3. Consider the function $f(x, y)$. The following information about f is given:

   - $f(2, 3) = 1$.
   - If you start at the point $(2, 3)$ and move towards the point $(1, 4)$, the directional derivative is 1.
   - Starting at $(2, 3)$ and moving towards $(0, 2)$ gives a directional derivative of $\sqrt{10}$.

   Find the following quantities:

   (a) $(\nabla f)(2, 3)$.

   (b) An approximate value for $f(2.1, 2.9)$.

OVER

4. The temperature at a point (x, y, z) is given by

$$T(x, y, z) = x^2 - y\cos(z) + e^{2z}.$$

(a) Find the rate of change of temperature at the point $P = (-1, 3, 0)$ in the direction of the vector $\vec{u} = \vec{i} - 2\vec{j} + 2\vec{k}$.

(b) In which direction does T change most rapidly at P? Find the rate of change of temperature in this direction.

(c) In heat conduction an *isotherm* is a surface on which temperature is constant. Find the equation of the isotherm which contains the point P. Determine the equation of the tangent plane to that surface at the point P.

5. (a) Use double or triple integrals to calculate the volume of the unit hemisphere.

(b) A cake shown on the right is in the shape of a unit hemisphere. The density of the horizontal layers of the cake varies with the distance from the base of the cake. The density of the base is 1. The density at the top is 0. The horizontal layer at the distance z from the base has the density $1 - z$. Calculate the mass of this cake.

(c) Comment on whether your answers in (5a) and (5b) are consistent. Explain your claim.

For your convenience I included a computer generated picture of such a cake and its cross section.

6. Let $A = (1, 1), B = (4, 1)$ and $C = (3, 2)$. Consider the house whose foundation is the triangle ABC and whose roof is given by $f(x, y) = x$.

(a) Give a simple underestimate and a simple overestimate for the volume of this house.

(b) Calculate the volume of this house.

(c) Make sure that your answers in (6a) and (6b) are consistent.

7. Find the minimum and maximum values of the function $f(x, y) = 2x^2 - y + y^2$ subject to the constraint $g(x, y) = 4x^2 + y^2 \leq 4$.

8. Let $A = (1, -1, 4)$, $B = (2, 0, 1)$ and $C = (0, 2, 3)$ be given points. As usual $O = (0, 0, 0)$ is the origin of the coordinate system.

(a) Find a unit vector $\vec{n}$ which is orthogonal to the plane that passes through the points A, B and C.

(b) Find the equation of the plane that passes through the points A, B, C.

(c) Find the area of the triangle ABC.

(d) Find value of scalar t such that the vector $t\,\vec{n} - \overrightarrow{OA}$ is orthogonal to the vector $\vec{n}$.

(e) Based on the previous item you can easily find the distance of the plane that passes through the points A, B, C from the origin. What is that distance?

① $V = \frac{r^2\theta}{2} * z = 1$ $\boxed{1}$

$$z = \frac{2}{r^2\theta}$$

$$S = 2rz + zr\theta + r^2\theta$$

$$= rz(2+\theta) + r^2\theta$$

$$S(r,\theta) = \frac{2}{r\theta}(2+\theta) + r^2\theta$$

$$S(r,\theta) = \frac{4}{r\theta} + \frac{2}{r} + r^2\theta$$

$$\frac{\partial S}{\partial r} = -\frac{4}{r^2\theta} - \frac{2}{r^2} + 2r\theta = 0$$

$$\frac{\partial S}{\partial \theta} = -\frac{4}{r\theta^2} + \cancel{2}r^2\cancel{\theta} = 0$$

$$\rightarrow r^3\theta^2 = 4$$

$$\xrightarrow{*r^2\theta} 2\underbrace{r^3\theta^2}_{=4} = 4 + 2\theta$$

$8 = 4 + 2\theta$

$\boxed{\theta = 2}$

$r^3\theta^2 = 4$

$\boxed{r = 1}$, $\boxed{z = 1}$

$$S(1,2) = \frac{2}{1\cdot 2}(2+2) + 1^2\cdot 2$$

$$= 6$$

(2) $\boxed{2}$

$$x = 2s \quad , y = 3t$$

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

$$9x^2 + 4y^2 = 36$$

$$9(2s)^2 + 4(3t)^2 = 36$$

$$9 \cdot 4 \cdot s^2 + 4 \cdot 9 \cdot t^2 = 36$$

$$\boxed{s^2 + t^2 = 1} \quad T$$

$$V = \iint_R (3+xy)\,dx\,dy = \iint_T (3+6st) \cdot 6\,ds\,dt$$

$$= 18 \iint_T (1+2st)\,ds\,dt$$

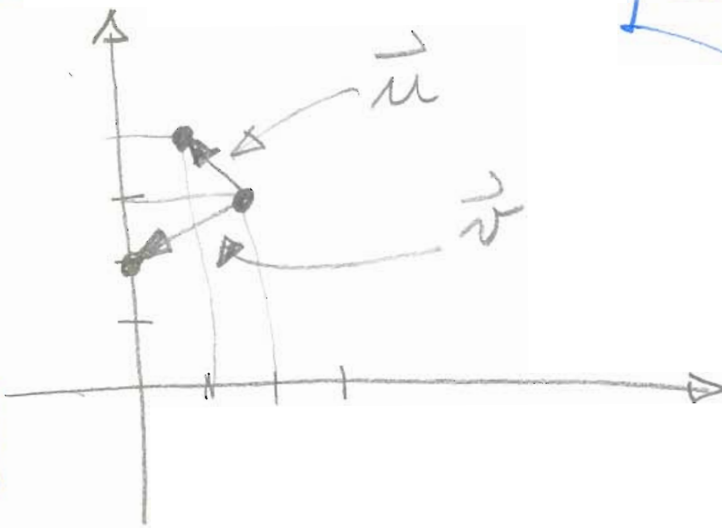
$$= 18 \underbrace{\iint_T ds\,dt}_{\pi} + 2 \underbrace{\iint_T st\,ds\,dt}_{0 \text{ by symmetry}}$$

$$= 18\pi = 18\pi$$

3 $f(2,3)=1$ $\boxed{3}$

$$\vec{u} = -\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$$

unit vector



$$\vec{v} = -\frac{2}{\sqrt{5}}\vec{i} - \frac{1}{\sqrt{5}}\vec{j}$$

unit vector

$$(\vec{\nabla} f)(2,3) = \underbrace{\frac{\partial f}{\partial x}(2,3)}_{a}\vec{i} + \underbrace{\frac{\partial f}{\partial y}(2,3)}_{b}\vec{j}$$

$$a\vec{i} + b\vec{j}$$

$$(a\vec{i}+b\vec{j})\cdot\left(-\frac{1}{\sqrt{2}}\vec{i}+\frac{1}{\sqrt{2}}\vec{j}\right) = 1$$

$$(a\vec{i}+b\vec{j})\cdot\left(-\frac{2}{\sqrt{5}}\vec{i}-\frac{1}{\sqrt{5}}\vec{j}\right) = \sqrt{10}$$

$$\Rightarrow -a+b=\sqrt{2}$$

$$-2a-b=\sqrt{10}\cdot\sqrt{5}=5\sqrt{2}$$

$$-3a = 6\sqrt{2}$$

$$\boxed{a=-2\sqrt{2} \quad b=-\sqrt{2}}$$

ⓐ $\boxed{\vec{\nabla} f = -\sqrt{2}(-2\vec{i}-\vec{j})}$

ⓑ $f(2.1, 2.9) \approx 1 - 2\sqrt{2}*0.1 + \sqrt{2}*0.1$

$= 1 - \sqrt{2}*0.1 \approx 1-0.141$

(4) $P = (-1, 3, 0)$

$\boxed{4}$

$$\frac{\partial T}{\partial x} = 2x, \quad \frac{\partial T}{\partial y} = -\cos z$$

$$\frac{\partial T}{\partial z} = y \sin z + 2e^{2z}$$

@ P $\quad \frac{\partial T}{\partial x}(-1,3,0) = -2$

$$\frac{\partial T}{\partial y}(-1,3,0) = -1$$

$$\frac{\partial T}{\partial y}(-1,3,0) = 2$$

~~unit vect~~ $(\vec{\nabla} T)(P) = -2\vec{i} - \vec{j} + 2\vec{k}$

(a) Directional derivative

$$(\vec{\nabla} T)(P) \cdot \underbrace{\frac{\vec{u}}{\|\vec{u}\|}}_{=3} = \frac{1}{3}\left((-2)\cdot 1 + (-1)\cdot(-2) + 2\cdot 2\right)$$

$$= \frac{1}{3}(\cancel{-2} + \cancel{2} + 4)$$

$$= 4/3$$

4 b The temperature changes most rapidly in the direction

$$-2\vec{i}-\vec{j}+2\vec{k}$$

The rate of change in this direction is $\sqrt{4+1+4}=3$.

c $T(-1,3,0)=1-3+1=-1$

$$(-2)x+(-1)y+2z=$$

$$=\underbrace{(-2)(-1)}_{2}+\underbrace{(-1)\cdot 3}_{(-3)}+\underbrace{2\cdot 0}_{0}$$

$$\boxed{-2x-y+2z=-1}$$ the tangent plane

or $\boxed{2x+y-2z=1}$

5

(5) (a) Use spherical coordinates $\boxed{6}$

$$\int_0^{2\pi}\int_0^1\int_0^{\pi/2} \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta =$$

$$= 2\pi \underbrace{\int_0^1 \rho^2 d\rho}_{1/3} \underbrace{\int_0^{\pi/2} \sin\phi \, d\phi}_{1}$$

$$= 2\pi/3$$

(b) $$\int_0^{2\pi}\int_0^1\int_0^{\pi/2} (1-\rho\cos\phi)\rho^2 \sin\phi \, d\phi \, d\rho \, d\theta$$

$$= \frac{2\pi}{3} - 2\pi \underbrace{\int_0^1 \rho^3 d\rho}_{1/4} \int_0^{\pi/2} \frac{1}{2}\sin 2\phi \, d\phi$$

$$= \frac{2\pi}{3} - \frac{\pi}{2}\cdot\frac{1}{2} \underbrace{\int_0^{\pi/2} \sin 2\phi \, d\phi}_{\left(-\frac{1}{2}\cos 2\phi\right)\Big|_0^{\pi/2} = \frac{1}{2}+\frac{1}{2} = 1}$$

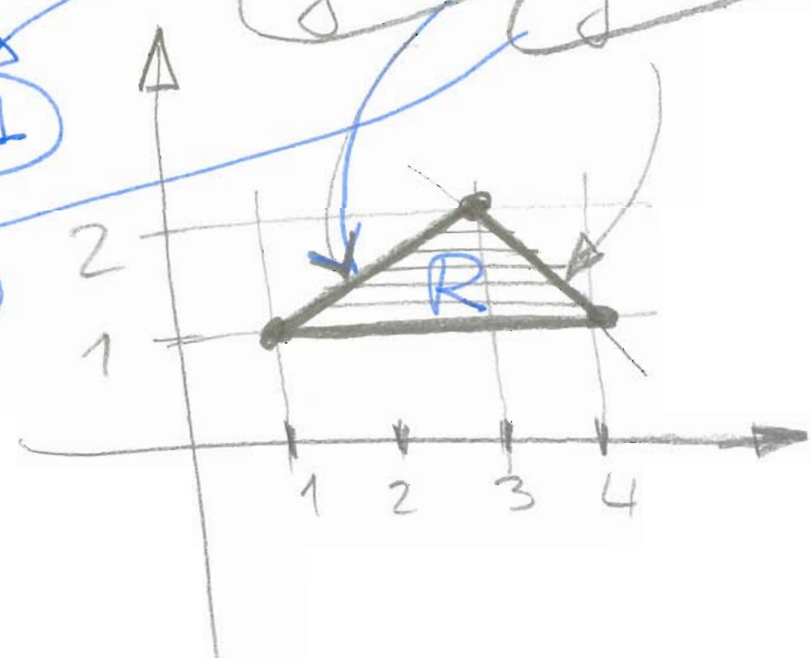
$$= \frac{2\pi}{3} - \frac{\pi}{4} = \frac{5\pi}{12}$$

5 c The number in b must be < than the number in a since in a we in fact assume that the density is 1 at each layer.

$y=\frac{1}{2}x+\frac{1}{2}$ $\quad$ $y=-x+5$

$x=2y-1$

6 b $x=-y+5$



$$\iint_R x\,dA =$$

$$= \int_1^2 \left(\int_{2y-1}^{-y+5} x\,dx \right) dy =$$

$$= \frac{1}{2} \int_1^2 x^2 \Big|_{2y-1}^{-y+5} dy =$$

7

$\boxed{8}$

$$= \frac{1}{2}\int_1^2 \left[(5-y)^2-(2y-1)^2\right] dy$$

$$= \frac{1}{2}\int_1^2 \left(25-10y+y^2-4y^2+4y-1\right) dy$$

6

$$= \frac{1}{2}\int_1^2 \left(24-6y-3y^2\right) dy$$

$$= \frac{1}{2}\left(24-3y^2\Big|_1^2-y^3\Big|_1^2\right)$$

$$-3\underbrace{(4-1)}_{3} - \underbrace{(8-1)}_{7}$$

$$24-9-7$$

$$24-16$$

$$= \frac{1}{2}\, 8 = 4$$

6 a Estimates: $\boxed{9}$

the lowest point of the roof is 1 the heighest point of the roof is 4. The area of the foundation is 3/2. Thus the volume is

$$\frac{3}{2} < V < 6$$

$$\frac{3}{2} < 4 < 6$$

Yes!

7 $\boxed{10}$

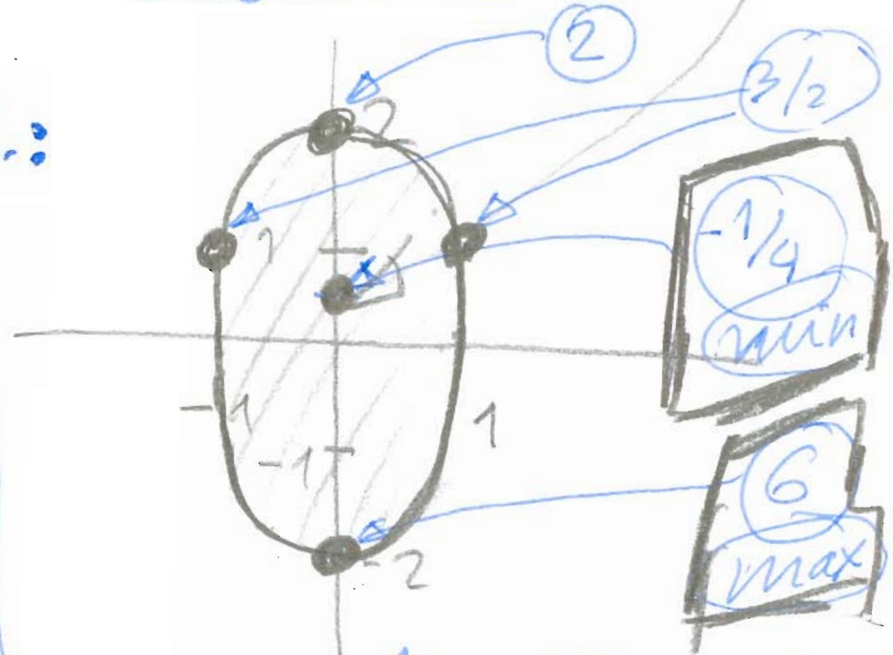
$$\frac{\partial f}{\partial x} = 4x = 0$$

$$\frac{\partial f}{\partial y} = -1 + 2y = 0$$

$$\boxed{f(0, 1/2) = -1/4} \qquad \boxed{\begin{matrix} x = 0 \\ y = 1/2 \end{matrix}}$$

Lagrange multipl.:

$$\boxed{\begin{aligned} 4x &= \lambda\, 8x \\ -1+2y &= \lambda\, 2y \\ 4x^2+y^2 &= 4 \end{aligned}}$$



$\lambda = 1/2$ OR $x = 0$

$x = 0 \quad y = \pm 2 \qquad (0,2),\ (0,-2)$

$f(0,2) = 2 \quad f(0,-2) = 6$

$\lambda = 1/2$

$-1 + \cancel{2}y = \cancel{y}$

$\boxed{y = +1}$

$x = \pm\sqrt{3/4} = \pm \frac{1}{2}\sqrt{3}$

$f\left(\pm\frac{\sqrt{3}}{2}\ 1\right) = \cancel{2}\frac{3}{4} - 1 + 1$

$= 3/2$

8 a

11

$$\vec{AB} = \vec{i} + \vec{j} - 3\vec{k}$$

$$\vec{AC} = -\vec{i} + 3\vec{j} - \vec{k}$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix}$$

$-1-3$

$$= 8\vec{i} + 4\vec{j} + 4\vec{k}$$

$$\|\vec{AB} \times \vec{AC}\| = 4\sqrt{2^2+1^2+1^2} = 4\sqrt{6}$$

$$\vec{n} = \frac{2}{\sqrt{6}}\vec{i} + \frac{1}{\sqrt{6}}\vec{j} + \frac{1}{\sqrt{6}}\vec{k}$$

b

$$\boxed{8x+4y+4z = \overset{}{8\cdot 0} + \overset{8}{4\cdot 2} + \overset{12}{4\cdot 3} = 20}$$

c $2\sqrt{6}$

8 d

$$t\vec{n} - \vec{OA} \perp \vec{n}$$

$$t(\vec{n} \cdot \vec{n}) - \vec{OA} \cdot \vec{n} = 0$$

$$t = \vec{OA} \cdot \vec{n}$$

$$= (\vec{i} - \vec{j} + 4\vec{k}) \cdot \left(\frac{2}{\sqrt{6}}\vec{i} + \frac{1}{\sqrt{6}}\vec{j} + \frac{1}{\sqrt{6}}\vec{k}\right)$$

$$= \frac{2}{\sqrt{6}} - \frac{1}{\sqrt{6}} + \frac{4}{\sqrt{6}} = \frac{5}{\sqrt{6}}$$

e The distance is $\frac{5}{\sqrt{6}}$.