MATH 224 Examination 4
March 2, 2012

Name ____________________

GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS. EACH PROBLEM IS WORTH 25 POINTS.

1. Consider the function $f(x, y) = x^3 - y$ and the constraint $x^2 + y^2 = 1$. Use Lagrange multipliers to find the minimum and maximum value of f subject to the given constraint. Proceed by answering all items below.

(a) Set up the system of equations for x, y and λ which will give all the candidate points for optimization of f under the given constraint. (You do not have to solve this system.)

(b) Figure 1 shows contours of f and the constraint. Use this figure to determine the number of solutions of the system in (1a). State this number clearly. Mark each point corresponding to a solution of the system in (1a) on Figure 1 by $P_1, P_2, \ldots$.

(c) Make a table with five columns:

1. the name of the point given in (1b) and marked on Figure 1,

2. approximate values of the x- and y-coordinates,

3. is $\lambda > 0$ or $\lambda < 0$,

4. state whether the point is local min or max,

5. is it global or local extreme.

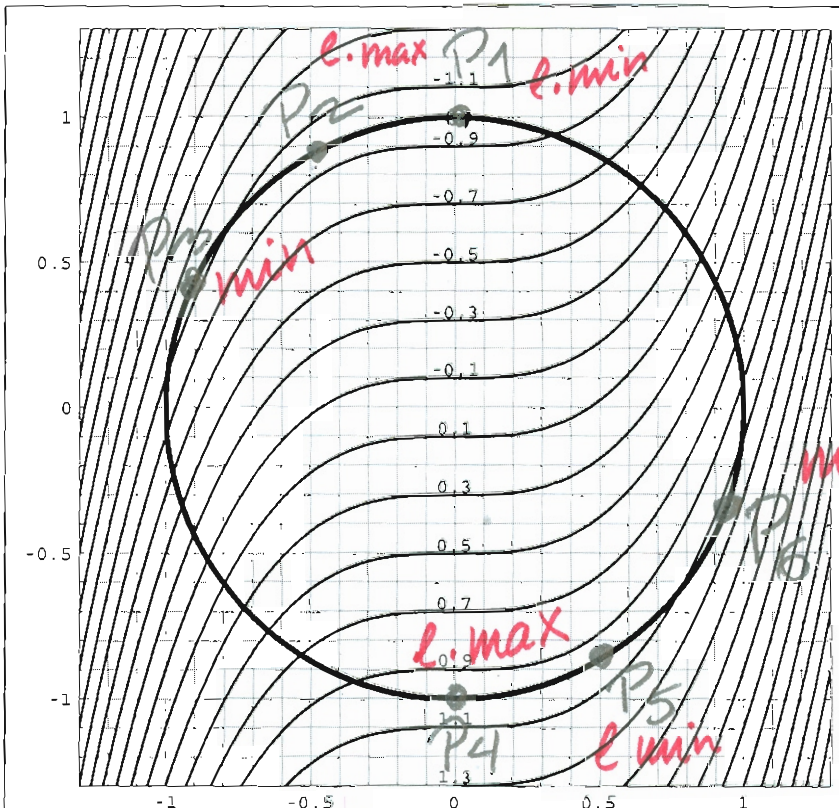

Figure 1

2. Use Lagrange multipliers to find the minimum and the maximum value of

$$f(x, y) = 2 - 2x^2 + y - y^2$$

subject to the constraint

$$x^2 + y^2 \leq 1.$$

Mark all relevant points that you calculated on the unit disc on the right.

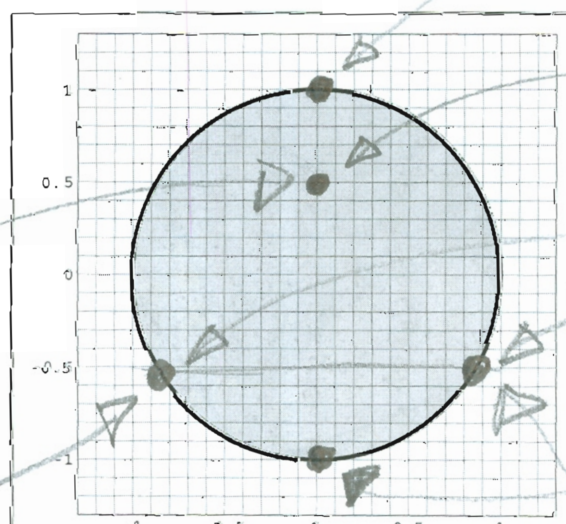

Figure 2

2

9/4

-1/4

max

0

min
global

min
global

OVER

① ⓐ

$$3x^2 = \lambda 2x$$
$$-1 = \lambda 2y$$
$$x^2 + y^2 = 1$$

$\boxed{1}$

Solving for x, y, λ gives all possible local extrema.

ⓑ There are 6 solutions to the above system. Two are easy

P_1 : $x = 0, y = -1, \lambda = 1/2$

P_2 : $x = 0, y = 1, \lambda = -1/2$

ⓒ

Point	(x,y)	λ	local	Global	f
P_1	$(0,1)$	<0	min	no	-1
P_2	$(-.5,.85)$	<0	max	no	$-.95$
P_3	$(-.9,.4)$	<0	min	min	-1.15
P_4	$(0,-1)$	>0	max	no	1
P_5	$(0.5,-.85)$	>0	min	no	0.95
P_6	$(.9,-.4)$	>0	max	max	1.15

① Inside: look for CPs $\boxed{2}$

$-4x = 0 \quad 1-2y = 0$

$x = 0 \quad y = 1/2$

Second derivative test

$f_{xx} = -4 \quad f_{xy} = 0 \quad f_{yy} = -2$

$f_{xx} f_{yy} - f_{xy}^2 = 8 > 0$

$f_{xx} < 0$ the point is local max

$f(0, 1/2) = 2 + \frac{1}{2} - \frac{1}{4} = 2 + \frac{1}{4} = 9/4$

Lagrange mult. ~~on the~~ with the constraint $x^2+y^2=1$

$-4x = \lambda 2x$

$1-2y = \lambda 2y$

$x^2+y^2 = 1$

$x = 0$

$y = \pm 1$

$\lambda = \cancel{\pm} 1/2$

$\lambda = -1/2, -3/2$

$x \neq 0 \quad \lambda = -2$

$1-2y = -4y, \; y = -1/2$

$y = -\frac{1}{2}, \; x = \pm \frac{\sqrt{3}}{2}$

3 $\qquad \displaystyle\int_1^2 \left(\int_0^{3-y} (3-x-y)\,dx \right) dy \qquad \boxed{3}$

$$= \int_1^2 \left((3-y)(3-y) - \frac{1}{2}(3-y)^2 \right) dy$$

$$= \frac{1}{2} \int_1^2 (3-y)^2\, dy =$$

$$= \frac{1}{2} \, \frac{-1}{3} (3-y)^3 \Big|_1^2$$

$$= -\frac{1}{6} * 1 + \frac{1}{6} 8 = \frac{7}{6}$$

4

$\boxed{4}$

$$\int_{-1}^{1}\left(\int_{-1}^{1}\left(\int_{0}^{\sqrt{2-x^2-y^2}} z\,dz\right)dy\right)dx$$

$$= \frac{1}{2}\int_{-1}^{1}\left(\underbrace{\int_{-1}^{1}(2-x^2-y^2)\,dy}\right)dx$$

$$= \frac{1}{2}\int_{-1}^{1}\left(2(2-x^2) - \frac{1}{3}y^3\Big|_{-1}^{1}\right)dx$$

$$= \frac{1}{2}\int_{-1}^{1}\left(4-2x^2-\frac{2}{3}\right)dx$$

$$= \frac{1}{2}\int_{-1}^{1}\left(\frac{10}{3}-2x^2\right)dx = \frac{1}{2}\left(\frac{20}{3}-\frac{2}{3}x^3\Big|_{-1}^{1}\right)$$

$$= \frac{1}{2}\left(\frac{20}{3}-\frac{4}{3}\right) = \frac{8}{3}$$