MATH 225 Examination 1
July 13, 2015

Name Key

GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS. EACH PROBLEM IS WORTH 25 POINTS.

1. In this problem we consider the unit sphere.

   (a) Use the spherical coordinates to evaluate the volume of the unit sphere.

   (b) Consider a fixed diameter of the unit sphere. Find the average distance between a point in the unit sphere and the fixed diameter.

2. Consider the vector field given by the equation:

$$\vec{F}(x,y) = -\frac{y}{2}\vec{i} + \frac{1}{2}\vec{j}\ .$$

   (a) Sketch a graph of this vector field.

   (b) Consider the following three parametric curves:

   A. $x(t) = \frac{1}{2}\cos t,\ y(t) = \frac{1}{2}\sin t;$ B. $x(t) = -\frac{1}{8}t^2 - 4,\ y(t) = \frac{1}{2}t;$ C. $x(t) = -\frac{1}{8}t^2 - 4,\ y(t) = -\frac{1}{2}t.$

   Determine which one of these three curves is a flow line of the vector field $\vec{F}$. Give both geometric (i.e. illustrate on the graph of the vector field) and algebraic reason for your answer.

3. The parametric curve in Figure 1 is given by the equations

$$x(t) = 2(1-t^2), \qquad y(t) = t^3.$$

   Calculate the arc length of this curve between the points $(0,1)$ and $(2,0)$. Give easy upper and lower estimates for the calculated arc length.

4. The vase pictured in Figure 2 is obtained by rotating the curve $x = 2 + \sin z,\ y = 0, z = z, 0 \leq z \leq 2\pi$ (the black curve in Figure 2), around z-axis. I decorated the vase with a curve starting at the point $(2,0,0)$ and wrapping around the vase twice and ending at the point $(2,0,2\pi)$. This is the red curve in Figure 2.

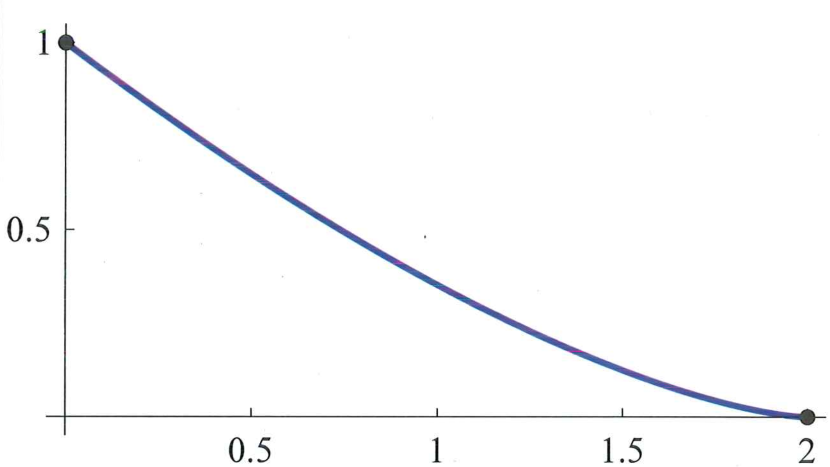


Figure 1: Calculate the arc length

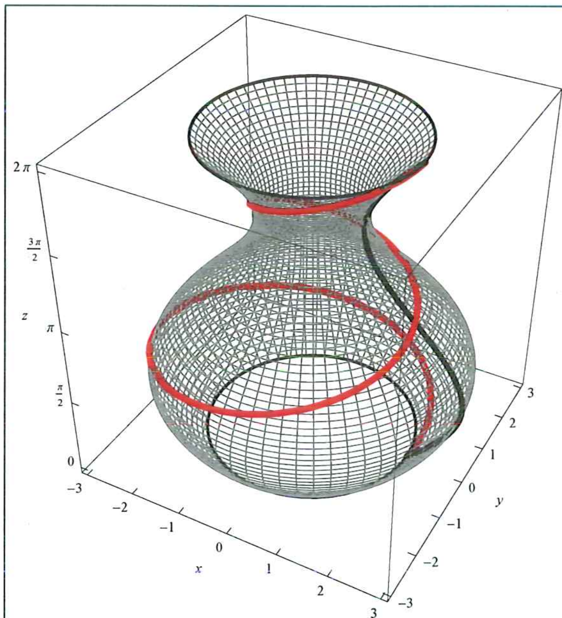


Figure 2: Find parametrizations

① ⓐ The volume of the unit sphere is given by the integral

$$\iiint_W 1 \cdot dV$$

where W is the unit sphere. The easiest way to describe the unit sphere algebraically is to use the polar coordinates:

$$W = \{(\rho, \theta, \phi) : 0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}.$$

So we proceed by calculating:

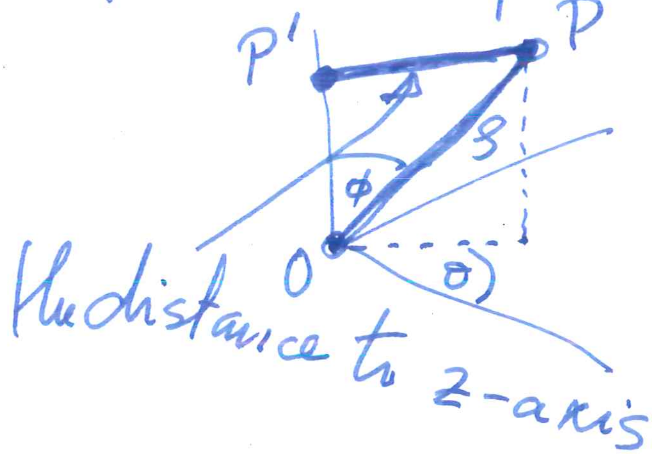
$$\iiint_W 1\, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{1} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \left(\frac{1}{3}\rho^3 \Big|_0^1\right) \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \left(\frac{1}{3} - 0\right) \sin\phi \, d\phi \, d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \underbrace{\left(\int_0^{\pi} \sin\phi \, d\phi\right)}_{=2} d\theta = \frac{2}{3}\int_0^{2\pi} d\theta = \frac{4\pi}{3}$$

ⓑ Here we use spherical coordinates again. Let (ρ, θ, ϕ) be the spherical coordinates of a point P. We need to calculate the distance from P to the z-axis.



$$\underbrace{\overline{PP'} = \rho\sin\phi}_{\text{the distance of } P \text{ to the z-axis}}$$

Now calculate

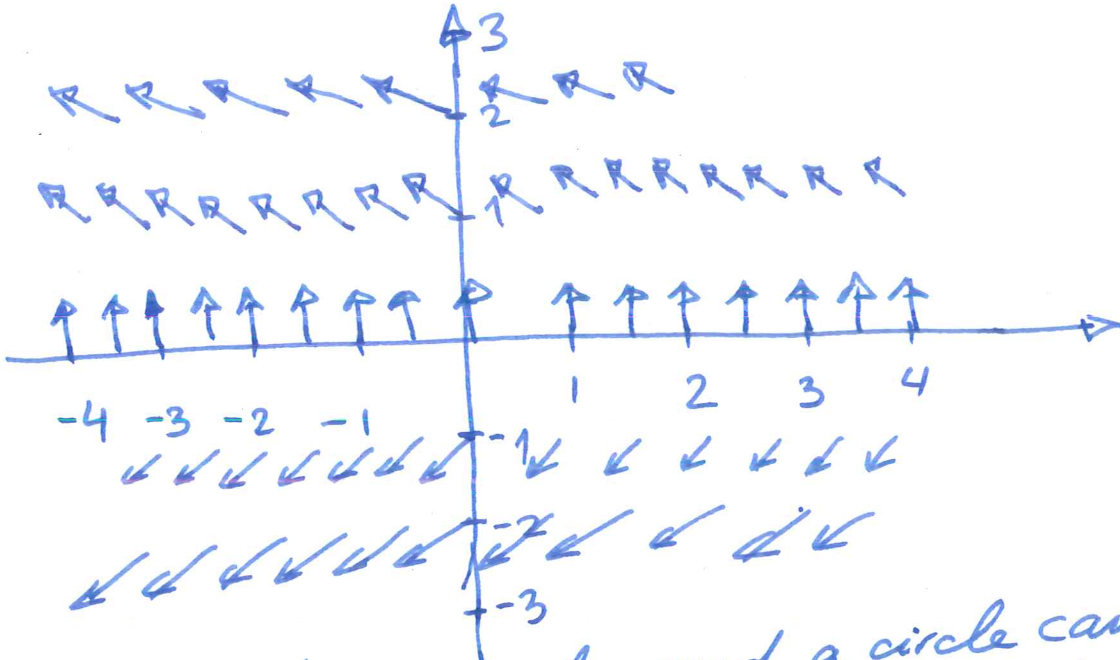
$$\iiint_W \rho\sin\phi\,\rho^2\sin\phi\,dV =$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^3(\sin\phi)^2\,d\rho\,d\phi\,d\theta =$$

$$= \frac{1}{4}\int_0^{2\pi}\underbrace{\int_0^{\pi}(\sin\phi)^2\,d\phi}_{\pi/2}\,d\theta = \frac{\pi}{8}\int_0^{2\pi}d\theta = \frac{\pi^2}{4}$$

Hence the average distance is $\dfrac{\frac{\pi^{\cancel{2}}}{4}}{\frac{4\pi}{3}} = \dfrac{3\pi}{16}$

2 a



b A represents a circle and a circle cannot be a flow line in the above vector field. The flow lines of the above field look like parabolas. Both B and C represent parabolas, but the particle described by C moves from up to down, so ~~the~~ a flow line is given by B. The algebraic reason for this is that for the functions given by B we have

$$x'(t) = -\frac{1}{4}t = -\frac{1}{2}y(t)$$

$$y'(t) = \frac{1}{2}$$

and this is exactly what is required from a flow line for $\vec{F}(x,y) = -\frac{y}{2}\vec{i} + \frac{1}{2}\vec{j}$

③ To calculate the length we need: $\boxed{4}$

$$x'(t) = -4t, \; y'(t) = 3t^2$$

thus $\vec{v}(t) = (-4t)\vec{i} + 3t^2\vec{j}$ and

$$\|\vec{v}(t)\| = \sqrt{16t^2 + 9t^4} = t\sqrt{16+9t^2}$$

For $t=0$ we get $(x(0), y(0)) = (2,0)$ and for $t=1$ we get $(x(1), y(1)) = (0,1)$.

The requested length is given by

$$\int_0^1 t\sqrt{16+9t^2}\,dt = \left| \begin{array}{l} u = 16+9t^2 \\ du = 18t\,dt \\ \begin{array}{c|c} t & u \\ \hline 0 & 16 \\ 1 & 25 \end{array} \end{array} \right| = \frac{1}{18}\int_{16}^{25} \sqrt{u}\,du$$

$$= \frac{1}{18}\,\frac{2}{3}u^{3/2}\Big|_{16}^{25} = \cancel{\frac{1}{18}}\,\frac{1}{27}\left(25^{3/2} - 16^{3/2}\right)$$

$$= \frac{1}{27}(125-64) = \frac{61}{27} \approx 2.25926$$

A lower estimate is given by $\sqrt{1+2^2} = \sqrt{5} \approx 2.23607$

An upper estimate is given by the tangent to the graph at $(0,1)$ and the x-axis.

The tangent vector at $(0,1)$ is $\boxed{5}$

$$x'(1)\vec{\imath} + y'(1)\vec{\jmath} = -4\vec{\imath} + 3\vec{\jmath}$$

The line ~~from~~ through $(0,1)$ in the direction of $\langle -4, 3\rangle$ is $\langle 0-4t, 1+3t\rangle$. This line crosses the x-axis at $t=-\frac{1}{3}$ at the point $(4/3, 0)$. The length of the line segment in the first quadrant is $\sqrt{1+\left(\frac{4}{3}\right)^2} = \sqrt{1+\frac{16}{9}} = \frac{5}{3}$. The length along the x-axis is $\frac{2}{3}$. Thus the upper estimate is $7/3 \approx 2.3333$

(4) The parametric equation of the vase is

$\langle (2+\sin z)\cos\theta, (2+\sin z)\sin\theta, z\rangle$ where $0 \le \theta \le 2\pi$, $0 \le z \le 2\pi$.

The red line circles twice around

the vase while climbing from $\boxed{6}$

0 to 2π. It is represented by

makes it circle twice

$$\left\langle (2+\sin z)\cos(2z),\ (2+\sin z)\sin(2z),\ z \right\rangle$$

$$0 \leq z \leq 2\pi$$

or

$$\left\langle \left(2+\sin\left(\frac{\theta}{2}\right)\right)\cos\theta,\ \left(2+\sin\left(\frac{\theta}{2}\right)\right)\sin\theta,\ \frac{\theta}{2} \right\rangle$$

$$0 \leq \theta \leq 4\pi$$

makes it climb from 0 to 2π while θ goes from 0 to 4π.