MATH 226 Examination 1 April 28, 2009

Name Key

Give all details of your reasoning. Each problem is worth 25 points for the total of 100 points.

Problem 1. (a) Write without the absolute values the exact value of the expression

$$\left|\pi^e - e^\pi\right|.$$

(b) Write the following English sentence as an inequality involving absolute value:

The distance between a number x and the number $-\frac{2}{3}$ is less than $\frac{1}{4}$.

Illustrate with a diagram on the number line.

Problem 2. (a) State the definition of the absolute value function.

(b) State all the properties of absolute value that you will need in (c). (No proofs are required, just the statements. You can not list any version of the triangle inequality here.)

(c) Prove that $\left|a+b\right| \le \left|a\right| + \left|b\right|$ for all $a, b \in \mathbb{R}$.

Problem 3. (a) State the definition of

$$\lim_{x\to+\infty} f(x) = L\ .$$

(b) Use the definition of limit to prove that

$$\lim_{x\to+\infty} \frac{x}{x+\cos x} = ?\ .$$

Problem 4. (a) State the ϵ-δ definition of continuity of a function f at a point a.

(b) Use ϵ-δ definition of continuity to prove that the function

$$f(x) = \frac{1}{x^2}$$

is continuous on $(0, +\infty)$.

(1a) Which one is bigger π^e or e^π. $\boxed{1}$

My guess is that $e^\pi > \pi^e$. But, this can be checked using calculator. Another way to prove this without using a calculator is to consider the function

$$f(x) = e^x - x^e, \quad x > 0$$

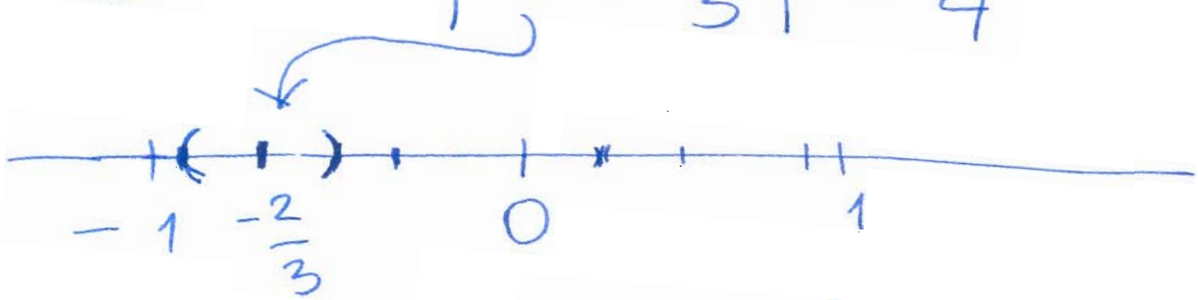
and <u>prove</u> that this function has the global <u>minimum</u> at $x = e$ with the value $f(e) = 0$. Thus $f(x) \geq 0$ whenever $x \neq e$. Consequently $e^\pi > \pi^e$, since $f(\pi) > 0$. Therefore

$$|\pi^e - e^\pi| = e^\pi - \pi^e.$$

this guess is reasonable since it is more efficient to use the bigger number as an exponent, rather than a base.

(1b)

$$\left| x + \frac{2}{3} \right| < \frac{1}{4}$$



(2a)

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

(2b) (1) $|u| = \max\{u, -u\}$ $\boxed{2}$

(2) $|u| \geq u$

(3) $|u| \geq -u$.

(2c) By (2b2) $a \leq |a|$

$b \leq |b|$

Hence $a+b \leq |a|+|b|$ (1)

By (2b3) $-a \leq |a|$

$-b \leq |b|$

Hence $-(a+b) \leq |a|+|b|$ (2)

By (2b?), (2c1) and (2c2) we deduce

$\max\{a+b, -(a+b)\} \leq |a|+|b|$

By (2b1) $|a+b| \leq |a|+|b|$.

(3a)

$$\lim_{x \to +\infty} f(x) = L$$

$\boxed{3}$

(I) There exists $X_0 \in \mathbb{R}$ such that $f(x)$ is defined for all $x \geq X_0$

(II) For each $\varepsilon > 0$ there exists $X(\varepsilon) \geq X_0$ such that

$$x > X(\varepsilon) \Rightarrow |f(x) - L| < \varepsilon .$$

(3b) (I) $X_0 = 2$. For $x \geq 2$ $x + \cos x > 0$, hence $\frac{x}{x+\cos x}$ is defined.

Next solve

$$\left| \frac{x}{x+\cos x} - 1 \right| < \varepsilon \text{ for } x .$$

$$\left| \frac{x}{x+\cos x} - 1 \right| \overset{A}{=} \left| \frac{x - x - \cos x}{x + \cos x} \right| \overset{AM}{\underset{A}{=}} \frac{|\cos x|}{|x + \cos x|}$$

$$\overset{AM}{=} \frac{|\cos x|}{x + \cos x} \underset{PP}{\leq} \frac{1}{x-1} .$$

Solve $\frac{1}{x-1} < \varepsilon$. Since $x - 1 > 0$

$$\frac{1}{\varepsilon} < x - 1$$

so $x > \frac{1}{\varepsilon} + 1$. Thus $X(\varepsilon) = \max\left\{\frac{1}{\varepsilon} + 1, 2\right\}$

(4a) A function f is continuous at a if the following two conditions are satisfied: $\boxed{4}$

(I) There exists $\delta_0 > 0$ such that $f(x)$ is defined for all $x \in (a-\delta_0, a+\delta_0)$.

(II) For every $\varepsilon > 0$ there exists $\delta(\varepsilon)$ such that $0 < \delta(\varepsilon) \leq \delta_0$

and

$$|x-a| < \delta(\varepsilon) \Rightarrow |f(x)-f(a)| < \varepsilon.$$

(4b) Let $a > 0$ be arbitrary.

(I) set $\delta_0 = \frac{a}{2}$. Clearly $1/x^2$ is defined for $x \in (a/2, 3a/2) \subseteq (0, +\infty)$.

(II) solve $\left|\frac{1}{x^2} - \frac{1}{a^2}\right| < \varepsilon$ for $|x-a|$.

$$\left|\frac{1}{x^2} - \frac{1}{a^2}\right| \underset{A}{=} \left|\frac{a^2-x^2}{x^2 \ast a^2}\right| \underset{\substack{A\\AM}}{=} \frac{|a^2-x^2|}{x^2a^2} \overset{A}{\underset{AM}{=}}$$

$$= \frac{|x-a|\,|x+a|}{x^2a^2} \underset{PP}{\leq} |x-a| \frac{|x+a|}{\frac{a^2}{4}a^2}$$

$$\underset{AM}{=} |x-a| \frac{x+a}{a^4/4} \underset{PP}{\le} |x-a| \frac{\frac{3a}{2}+a}{\frac{a^4}{4}} \qquad \boxed{5}$$

$$= |x-a| \frac{\frac{5\cancel{a}}{\cancel{2}}}{\frac{a^{\cancel{4}3}}{\cancel{4}2}} = |x-a| \frac{10}{a^3}$$

Now solve $|x-a| \frac{10}{a^3} < \varepsilon$ to find

$|x-a| < \frac{a^3 \varepsilon}{10}$. Thus $\delta(\varepsilon) = \min\left\{\frac{a^3\varepsilon}{10}, \frac{a}{2}\right\}$.