ON TWO COMMON SEQUENCES

BRANKO ĆURGUS

In this note we shall give a simple and easy-to-remember proof that two sequences ($\{a_n\}$ and $\{s_n\}$ defined below), commonly used to define the number e, converge to the same limit. Surprisingly, many elementary analysis textbooks do not include this topic. The proofs in the classical book [2, Theorem 3.31] and in a more recent book [1, Proposition 3.3.1] are more involved. Related questions have been considered in [3] and [4], however.

We start by recalling Bernoulli's inequality. It states that for all real numbers r with $r > -1, r \neq 0$, and all integers m greater than 1,

$$(1+r)^m > 1 + rm.$$

We also recall that the binomial theorem states that for all real numbers x and y, and all positive integers m,

$$(x+y)^m = \sum_{k=0}^{m} \binom{m}{k} x^{m-k} y^k,$$

where $\binom{m}{k} = \frac{m!}{k!(m-k)!}$ are binomial coefficients.

By $\mathbb{N}$ we denote the set of all positive integers. The following two sequences are commonly used to define the number e:

$$a_n = \left(1 + \frac{1}{n}\right)^n, \qquad n \in \mathbb{N},$$

$$s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} = \sum_{k=0}^{n} \frac{1}{k!}, \quad n \in \mathbb{N}.$$

Proposition 1. *The sequence $\{s_n\}$ is increasing and bounded above by* 3.

Proof. The sequence $\{s_n\}$ is increasing since

$$s_{n+1} - s_n = \frac{1}{(n+1)!} > 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

Clearly $s_1 < 3$. Further, notice that $1/k! \leq 1/\big((k-1)k\big)$ for all integers k with $k \geq 2$. Therefore, for all integers n greater than 1 we have

$$\begin{aligned}
s_n &= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} + \frac{1}{n!} \\
&\leq 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-2)(n-1)} + \frac{1}{(n-1)n} \\
&= 2 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-2} - \frac{1}{n-1}\right) + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\
&= 3 - \frac{1}{n} \\
&< 3.
\end{aligned}$$

This proves that 3 is an upper bound for $\{s_n\}$. □

Proposition 2. *The following inequalities hold: $a_1 = s_1$ and for all integers n greater than* 1*,*

$$s_n - \frac{3}{n} < a_n < s_n. \tag{1}$$

Proof. A straightforward verification yields $a_1 = s_1$ and $s_2 - 3/2 < a_2 < s_2$. Now let n be an integer greater than 2. The following proof of (1) is a succession of five steps each suggesting the next one.

1. The binomial theorem with $x = 1$, $y = 1/n$ and $m = n$ yields

$$a_n = \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\frac{1}{n^k} = 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!}\,\frac{n!}{(n-k)!\,n^k}. \tag{2}$$

2. Let k be integer with $2 \leq k \leq n$. The coefficient of $1/k!$ in expansion (2) for a_n is, after cancellation of common terms in the numerator and denominator, a product of exactly k factors:

$$\begin{aligned}\frac{n!}{(n-k)!\,n^k} &= \frac{n(n-1)\cdots(n-k+1)}{n^k}\\ &= \left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{n-k+1}{n}\right)\\ &= 1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right).\end{aligned}$$

3. The number 1 is the greatest and $\big(1 - (k-1)/n\big)$ is the smallest factor of the last product. Therefore,

$$\left(1-\frac{k-1}{n}\right)^k < \frac{n!}{n^k(n-k)!} = 1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) < 1^k = 1.$$

4. Bernoulli's inequality with $r = -(k-1)/n$ and $m = k$ yields

$$\left(1-\frac{k-1}{n}\right)^k > 1 - k\,\frac{k-1}{n} = 1 - \frac{(k-1)k}{n}.$$

The last two displayed relations together imply

$$1 - \frac{(k-1)k}{n} < \frac{n!}{n^k(n-k)!} < 1. \tag{3}$$

5. Inequalities (3) give bounds for the coefficient of $1/k!$ in (2). The consequent inequalities for a_n are

$$1 + 1 + \sum_{k=2}^{n}\frac{1}{k!}\left(1-\frac{(k-1)k}{n}\right) < a_n < 1 + 1 + \sum_{k=2}^{n}\frac{1}{k!}\cdot 1 = s_n. \tag{4}$$

Finally, a simplification of the left-hand side of (4) shows that it is equal to

$$\sum_{k=0}^{n}\frac{1}{k!} - \sum_{k=2}^{n}\frac{1}{k!}\frac{(k-1)k}{n} = s_n - \frac{1}{n}\sum_{k=2}^{n}\frac{1}{(k-2)!} = s_n - \frac{1}{n}s_{n-2}. \tag{5}$$

Moreover, by Proposition 1 $s_{n-2} < 3$ and therefore $s_n - \frac{1}{n}s_{n-2} > s_n - 3/n$. Hence, the left-hand side of (4) is greater than $s_n - 3/n$. Thus, (1) holds for $n > 2$. □

Theorem 3. *The sequences $\{a_n\}$ and $\{s_n\}$ converge to the same limit.*

Proof. The sequence $\{s_n\}$ converges by the Monotone Convergence Theorem and Proposition 1. The sequence $\{a_n\}$ converges to the same limit by the Squeeze Theorem and Proposition 2. □

Theorem 3 justifies the following definition.

Definition 4. The number e is the common limit of the sequences $\{a_n\}$ and $\{s_n\}$.

References

[1] K. R. Davidson, A. P. Donsing, *Real Analysis with Real Applications.* Prentice Hall, 2002.

[2] W. Rudin, *Principles of Mathematical Analysis.* Third edition. International Series in Pure and Applied Mathematics. McGraw-Hill, 1976.

[3] N. Schaumberger, Another proof of the formula $e = \sum(1/n!)$, The College Math. Journal **25** (1994) 38-39.

[4] J. Wiener, Bernoulli's inequality and the number e, The College Math. Journal **16** (1985) 399-400.