Limits

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1 Numbers

All numbers in these notes are real numbers. The set of all real numbers is denoted by $\mathbb{R}$.

The most important subsets of real numbers are the set of natural numbers, denoted by $\mathbb{N}$, and the set of integers, denoted by $\mathbb{Z}$. That is

$$\mathbb{N} = \{1, 2, 3, \ldots\}, \qquad \mathbb{Z} = \{-n : n \in \mathbb{N}\} \cup \{0\} \cup \mathbb{N}.$$

Important subsets of $\mathbb{R}$ are intervals. Let a and b be real numbers such that $a < b$. Here are all possible intervals with endpoints a and b:

$$\begin{array}{llll} x \in [a,b] & \text{means} \;\; a \leq x \leq b, & x \in (a,b) & \text{means} \;\; a < x < b, \\ x \in [a,b) & \text{means} \;\; a \leq x < b, & x \in (a,b] & \text{means} \;\; a < x \leq b. \end{array}$$

The set $[a,b]$ is called a *closed interval*. The set (a,b) is called an *open interval*. The sets $[a,b)$ and $(a,b]$ are called *half-open interval* or *half-closed interval*. These intervals are called *finite intervals*. The *infinite intervals* are

$$\begin{array}{ll} [a,+\infty) := \bigl\{x \in \mathbb{R} : x \geq a\bigr\}, & (a,+\infty) := \bigl\{x \in \mathbb{R} : x > a\bigr\}, \\ (-\infty,a] := \bigl\{x \in \mathbb{R} : x \leq a\bigr\}, & (-\infty,a) := \bigl\{x \in \mathbb{R} : x < a\bigr\}. \end{array}$$

The set $\mathbb{R}$ is also an infinite interval. Sometimes it is written as $(-\infty,+\infty)$.

Let S be a subset of $\mathbb{R}$. If u is the smallest number in S, then u is called a *minimum* of S and we write $u = \min S$. If v is the greatest number in S, then v is called a *maximum* of S and we write $v = \max S$. Notice that the set $\mathbb{Z}$ has neither a minimum nor a maximum. Also (a,b) has neither a minimum nor a maximum. The set $\mathbb{N}$ has no maximum and $\min \mathbb{N} = 1$. Each finite subset of $\mathbb{R}$ has both a minimum and a maximum.

2 Functions

2.1 The definition

Next we review the definition of a function. Let A and B be sets. A *function* f from A to B is a rule that assigns **exactly one** element of B to **each** element in A. This relationship between the sets A and B and the rule f is indicated by the following notation: $f : A \to B$. For $x \in A$ the unique element of B which is assigned to x by the function f is called the value of f at x.

This element is denoted by $f(x)$. The set A is *domain* of f. The subset $\{f(x) \in B \ : \ x \in A\}$ of B is the *range* of f.

In this class we are interested in functions for which both sets A and B are subsets of the set of real numbers $\mathbb{R}$. Some examples of such functions are given below.

2.2 Examples

For each of the examples below answer the following questions: (a) What are the domain and the range of the function? (b) Plot the function using your graphing calculator. Plot the function by hand emphasizing the details missed by your graphing calculator.

Example 2.2.1. Let $\mathrm{Sign} : \mathbb{R} \to \mathbb{R}$ be given by the formula

$$\mathrm{Sign}(x) := \begin{cases} \ \ 1 & \text{for } \ x > 0, \\ \ \ 0 & \text{for } \ x = 0, \\ -1 & \text{for } \ x < 0. \end{cases}$$

This function is called the *sign* function.

Example 2.2.2. Let $\mathrm{UnitStep} : \mathbb{R} \to \mathbb{R}$ be given by the formula

$$\mathrm{UnitStep}(x) := \begin{cases} 1 & \text{for } \ x \geq 0, \\ 0 & \text{for } \ x < 0. \end{cases}$$

This function is called the *unit step* function.

Example 2.2.3. Let $\mathrm{Floor} : \mathbb{R} \to \mathbb{R}$ be given by the formula

$$\mathrm{Floor}(x) = \lfloor x \rfloor := \max\bigl\{k \in \mathbb{Z} : k \leq x\bigr\}.$$

This function is called the *floor* function. In other words for a given $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the unique integer with the following property

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1.$$

As an immediate consequence we get that

$$x - 1 < \mathrm{Floor}(x) \leq x \qquad \text{for all} \qquad x \in \mathbb{R}.$$

Example 2.2.4. Let $\mathrm{Ceiling} : \mathbb{R} \to \mathbb{R}$ be given by the formula

$$\mathrm{Ceiling}(x) = \lceil x \rceil := \min\{k \in \mathbb{Z} : k \geq x\}.$$

This function is called the *ceiling* function.

(a) Prove that $x \leq \mathrm{Ceiling}(x) < x + 1$ for all $x \in \mathbb{R}$.

Example 2.2.5. Let $\mathrm{Abs} : \mathbb{R} \to \mathbb{R}$ be given by the formula

$$\mathrm{Abs}(x) = |x| := \begin{cases} \ \ x & \text{if } \ x \geq 0, \\ -x & \text{if } \ x < 0. \end{cases}$$

This function is called the *absolute value* function.

Exercise 2.2.6. Prove that $\max\{u, v\} = v + (u - v)\,\mathrm{UnitStep}(u - v)$ for all $u, v \in \mathbb{R}$.

2.3 Absolute value

For a given real number a the number $|a|$ is called the *absolute value* of the number a.

From calculus you are familiar with the geometric representation of real numbers as points on a straight line. This is done by choosing a point on the line to represent 0 and another point to represent 1. Then, every real number will correspond to a point on this line (called the number line), and every point on the number line will correspond to a real number. This geometric representation might be very helpful in doing the problems.

Geometrically, the absolute value of a represents the distance between 0 and a, or, generally $|a-b|$ is the *distance* between the real numbers a and b on the number line.

The basic properties of the absolute value are given in the following exercises.

Exercise 2.3.1. Prove the following statements.

(i) $|x| = \max\{x, -x\}$.

(ii) $|x| \geq 0$ for all $x \in \mathbb{R}$.

(iii) $|-x| = |x|$ for all $x \in \mathbb{R}$.

(iv) $-x \leq |x|$ and $x \leq |x|$ for all $x \in \mathbb{R}$.

(v) $|xy| = |x||y|$ for all $x, y \in \mathbb{R}$.

(vi) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$ for all $x, y \in \mathbb{R}$, $y \neq 0$.

Proof. To prove (i) we consider two cases. **Case I.** Assume $x \geq 0$. Then $-x \leq 0$. Since $-x \leq 0$ and $0 \leq x$, it follows that $-x \leq x$. Therefore $\max\{x, -x\} = x$. By Definition in Example 2.2.5 for $x \geq 0$ we have that $\operatorname{Abs}(x) = x$. Hence, we conclude that $\operatorname{Abs}(x) = \max\{x, -x\}$ in this case. **Case II.** Assume $x < 0$. Then $-x > 0$. Since $-x > 0$ and $0 > x$, it follows that $-x > x$. Therefore $\max\{x, -x\} = -x$. By Definition in Example 2.2.5 for $x < 0$ we have that $\operatorname{Abs}(x) = -x$. Hence, we conclude that $\operatorname{Abs}(x) = \max\{x, -x\}$ in this case.

Since Cases I and II include all real numbers x, the equality $\operatorname{Abs}(x) = \max\{x, -x\}$ is proved.

The statement (ii) can also be proved by considering two cases.

To prove (iii) note that by (i) $|x| = \max\{x, -x\}$ and also $|-x| = \max\{-x, -(-x)\} = \max\{-x, x\}$. Since $\max\{x, -x\} = \max\{-x, x\}$, we conclude that $|x| = |-x|$.

By the definition of max we have $\max\{a, b\} \geq a$ and $\max\{a, b\} \geq b$ for any real numbers a and b. Therefore $\max\{x, -x\} \geq x$ and $\max\{x, -x\} \geq -x$. Using (i) we conclude $|x| \geq x$ and $|x| \geq -x$. This proves (iv). $\square$

Exercise 2.3.2. Let $x \in \mathbb{R}$ and $a > 0$. Prove that $|x| < a$ if and only if $-a < x < a$.

Exercise 2.3.3. (a) Let $a, b \in \mathbb{R}$. Prove that $|a + b| \leq |a| + |b|$.

(b) Let $x, y, z \in \mathbb{R}$. Prove that $|x - y| \leq |x - z| + |z - y|$.

(c) Let $x, y \in \mathbb{R}$. Prove that $\big||x| - |y|\big| \leq |x - y|$.

Proof. Proof of (a). By Exercise 2.3.1 (iv) we know that $a \leq |a|$ and $b \leq |b|$. Therefore we conclude that

$$a + b \leq |a| + |b|. \tag{2.3.1}$$

By Exercise 2.3.1 (iv) we know that $-a \leq |a|$ and $-b \leq |b|$. Therefore we conclude

$$-(a + b) = -a + (-b) \leq |a| + |b|. \tag{2.3.2}$$

The inequalities (2.3.1) and (2.3.2) imply

$$\max\{a+b, -(a+b)\} \leq |a| + |b|. \tag{2.3.3}$$

By Exercise 2.3.1 (i) the inequality (2.3.3) yields $|a+b| \leq |a| + |b|$. This proves (a).

Prove (b) and (c) as an exercise. □

The inequalities in Exercise 2.3.3 are often called the *Triangle Inequalities.*

2.4 New functions from old

Definition 2.4.1. Given two functions $f : A \to B$ and $g : A \to B$, with $A, B \subset \mathbb{R}$, and two real numbers α and β we form a new function $\alpha f + \beta g : A \to B$ defined by

$$(\alpha f + \beta g)(x) := a\, f(x) + \beta\, g(x), \quad \text{for all} \quad x \in A.$$

Notice that $f(x)$ and $g(x)$ are real numbers so that $\alpha\, f(x)$ and $\beta\, g(x)$ in the above formula is just a multiplication of real numbers. The function $\alpha f + \beta g$ is called a *linear combination* of the functions f and g.

Definition 2.4.2. Given two functions $f : A \to B$ and $g : A \to B$, with $A, B \subset \mathbb{R}$ we form a new function $fg : A \to B$ defined by

$$(fg)(x) := f(x)g(x), \quad \text{for all} \quad x \in A.$$

Notice that $f(x)$ and $g(x)$ are real numbers so that $f(x)g(x)$ in the above formula is just a multiplication of real numbers. The function fg is called the *product* of the functions f and g.

Definition 2.4.3. Given two functions $f : A \to B$ and $g : B \to C$ a new function $g \circ f : A \to C$ is defined by

$$(g \circ f)(x) := g(f(x)), \qquad x \in A.$$

The function $g \circ f$ is called the *composition* of the functions f and g.

Applying these definitions to familiar functions gives rise to new, sometimes very interesting functions.

2.5 More examples

Exercise 2.5.1. For each of the functions given below answer the following questions: (a) What are the domain and the range of the function? (b) Plot the function using your graphing calculator. Plot the function by hand emphasizing the details missed by your graphing calculator.

(a)	$x \mapsto x\,\mathrm{Abs}(x)$	(b)	$x \mapsto x(1 - \mathrm{Abs}(x))$
(c)	$x \mapsto x\,\mathrm{Sign}(x)$	(d)	$x \mapsto \mathrm{Ceiling}(x) - \mathrm{Floor}(x)$
(e)	$x \mapsto x - \mathrm{Floor}(x)$	(f)	$x \mapsto x\,\mathrm{Floor}(1/x)$
(g)	$x \mapsto (1 + \mathrm{Sign}(x))/2$	(h)	$x \mapsto x\ \mathrm{UnitStep}(x)$
(i)	$x \mapsto \mathrm{Sign}(\mathrm{Abs}(x))$	(j)	$x \mapsto \mathrm{Abs}(\mathrm{Sign}(x))$
(k)	$x \mapsto \mathrm{Floor}(\mathrm{Abs}(x))$	(l)	$x \mapsto \mathrm{Ceiling}(\mathrm{Abs}(x))$

3 Limit of a function as x approaches $+\infty$

3.1 The definition

Definition 3.1.1. A function $x \mapsto f(x)$ has the limit L as x approaches $+\infty$ if the following two conditions are satisfied:

(I) There exists a real number X_0 such that $f(x)$ is defined for each $x \geq X_0$.

(II) For each real number $\epsilon > 0$ there exists a real number $X(\epsilon) \geq X_0$ such that

$$x > X(\epsilon) \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

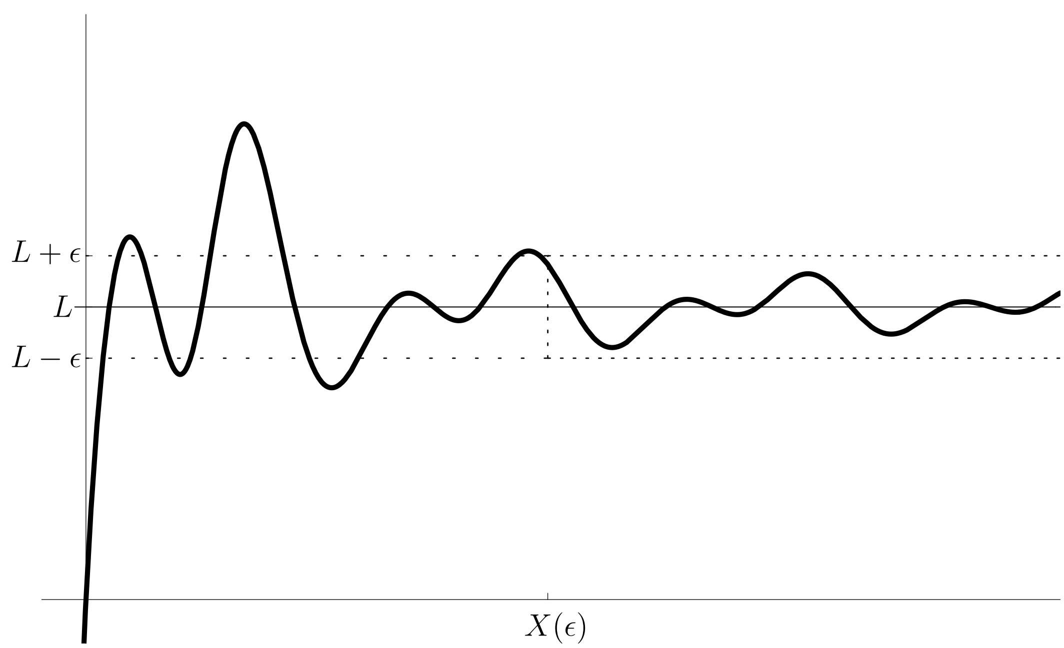


Figure 1: An illustration for (II) in Definition 3.1.1

If the conditions (I) and (II) in Definition 3.1.1 are satisfied we write $\lim_{x \to +\infty} f(x) = L$.

3.2 Examples for Definition 3.1.1

Example 3.2.1. Prove that $\lim_{x \to +\infty} \dfrac{1}{\sqrt{x-1}} = 0$.

Solution. We have to show that the conditions (I) and (II) in Definition 3.1.1 are satisfied. First we have to provide X_0. We can take $X_0 = 2$, since if $x \geq 2$, then $x - 1 > 0$ and $1/\sqrt{x-1}$ is defined.

Next we show that the condition (II) is satisfied. Let $\epsilon > 0$ be given. We have to find a real number $X(\epsilon) \geq 2$ such that

$$x > X(\epsilon) \quad \Rightarrow \quad \left| \frac{1}{\sqrt{x-1}} - 0 \right| < \epsilon. \tag{3.2.1}$$

In some sense we have to solve the inequality

$$\left| \frac{1}{\sqrt{x-1}} - 0 \right| < \epsilon.$$

for x. The first step is to simplify it. Clearly

$$\left| \frac{1}{\sqrt{x-1}} - 0 \right| = \frac{1}{\sqrt{x-1}} \quad \text{for} \quad x \geq 2.$$

Thus we need to solve

$$\frac{1}{\sqrt{x-1}} < \epsilon.$$

This inequality is solved for x by using the following sequence of algebraic steps:

$$\frac{1}{\sqrt{x-1}} < \epsilon \quad \Leftrightarrow \quad \sqrt{x-1} > \frac{1}{\epsilon} \quad \Leftrightarrow \quad x-1 > \frac{1}{\epsilon^2} \quad \Leftrightarrow \quad x > \frac{1}{\epsilon^2} + 1. \tag{3.2.2}$$

Since we need $X(\epsilon) \geq 2$, we choose $X(\epsilon) := \max\left\{ \frac{1}{\epsilon^2} + 1, 2 \right\}$.

It remains to prove that the implication (3.2.1) is satisfied. Assume that

$$x > X(\epsilon). \tag{3.2.3}$$

Since $X(\epsilon) \geq 2$, we conclude that $x > 2$. Therefore $x - 1 > 0$ and $1/\sqrt{x-1}$ is defined. Since $X(\epsilon) \geq 1/\epsilon^2 + 1$, we conclude that

$$x > \frac{1}{\epsilon^2} + 1.$$

Now the equivalences (3.2.2) imply that

$$\frac{1}{\sqrt{x-1}} < \epsilon. \tag{3.2.4}$$

Since $1/\sqrt{x-1}$ is positive we conclude that

$$\frac{1}{\sqrt{x-1}} = \left| \frac{1}{\sqrt{x-1}} \right| = \left| \frac{1}{\sqrt{x-1}} - 0 \right|. \tag{3.2.5}$$

Combining (3.2.4) and (3.2.5), yields

$$\left| \frac{1}{\sqrt{x-1}} - 0 \right| < \epsilon. \tag{3.2.6}$$

Thus, we have proved that the assumption (3.2.3) implies the inequality (3.2.6). This is exactly the implication (3.2.1). $\square$

Example 3.2.2. Determine the limit of the function $x \mapsto \dfrac{\text{Ceiling}(x)}{x}$ as x approaches $+\infty$ and prove your claim.

Solution. In Example 2.2.4 it is established that $x \leq \text{Ceiling}(x) < x + 1$ for each real number x. Therefore, for large x, the value of Ceiling(x) does not differ much from x. Therefore it is reasonable to make the following claim

$$\lim_{x \to +\infty} \frac{\text{Ceiling}(x)}{x} = 1.$$

Next we shall prove this claim using Definition 3.1.1. Since the function $x \mapsto \dfrac{\text{Ceiling}(x)}{x}$ is defined for all $x \neq 0$, we can take $X_0 = 1$.

Next we show that the condition (II) is satisfied. Let $\epsilon > 0$ be given. We have to find a real number $X(\epsilon) \geq 1$ such that

$$x > X(\epsilon) \quad \Rightarrow \quad \left| \frac{\text{Ceiling}(x)}{x} - 1 \right| < \epsilon. \tag{3.2.7}$$

Solving for x the inequality

$$\left| \frac{\text{Ceiling}(x)}{x} - 1 \right| < \epsilon \tag{3.2.8}$$

is not easy. To find solutions of this inequality we first need to simplify it. In this process of simplification we can replace the expression

$$\left| \frac{\text{Ceiling}(x)}{x} - 1 \right|$$

with an expression which is greater or equal to it. In Example 2.2.4 we learned that

$$x \leq \text{Ceiling}(x) < x + 1. \tag{3.2.9}$$

Since we consider only $x \geq 1$, we can divide by x in (3.2.9) and subtract 1 from each term to get

$$0 \leq \frac{\text{Ceiling}(x)}{x} - 1 < \frac{x+1}{x} - 1 = \frac{1}{x}.$$

Therefore

$$\left| \frac{\text{Ceiling}(x)}{x} - 1 \right| \leq \frac{1}{x} \qquad \text{for all} \qquad x \geq 1. \tag{3.2.10}$$

This inequality is the key step in this proof. Therefore I call it the BIg INequality, or BIN. (Each of the proofs involving the definition of limit involves a BIN.) The importance of BIN lies in the fact that instead of solving (3.2.8), we can solve for x the simpler inequality

$$\frac{1}{x} < \epsilon.$$

The solution of this inequality (remember $x \geq 1$) is $x > \dfrac{1}{\epsilon}$.

Now we can define $X(\epsilon) := \max\left\{\frac{1}{\epsilon}, 1\right\}$. With this $X(\epsilon)$ the implication (3.2.7) is true. It is easy to prove this claim: Assume that

$$x > X(\epsilon) = \max\left\{\frac{1}{\epsilon}, 1\right\}.$$

Then $x \geq 1$ and $x > \frac{1}{\epsilon}$. Since $x \geq 1$ the BIN inequality (see (3.2.10))

$$\left|\frac{\mathrm{Ceiling}(x)}{x} - 1\right| \leq \frac{1}{x}$$

is true. Since also $x > \frac{1}{\epsilon}$, we conclude that

$$\frac{1}{x} < \epsilon.$$

The last two displayed inequalities imply that

$$\left|\frac{\mathrm{Ceiling}(x)}{x} - 1\right| < \epsilon.$$

This proves the implication (3.2.7). $\square$

Exercise 3.2.3. Determine whether the following functions have limits as x approaches $+\infty$. Prove your statements using the definition.

(a) $x \mapsto \dfrac{x}{3x-2}$ (b) $x \mapsto \dfrac{2x}{x^2+x+1}$ (c) $x \mapsto \dfrac{x+\sin(x)}{x-1}$

(d) $x \mapsto \dfrac{x^2+x}{x^3+3}$ (e) $x \mapsto \dfrac{x^3-2x^2+1}{x^3+x+101}$ (f) $x \mapsto \sqrt{x+1} - \sqrt{x}$

(g) $x \mapsto \dfrac{x^2+x\cos(x)}{x^2-x+5}$ (h) $x \mapsto \left(\dfrac{1}{x}\right)^{1/\ln x}$ (i) $x \mapsto \dfrac{x^2-1}{x^2+2x\sin(x)}$

(j) $x \mapsto x - \sqrt{x^2-x}$

Exercise 3.2.4. Guess the limit of the function $x \mapsto \ln\left(1+\frac{1}{x}\right)^x$ and prove your guess.

Hint: 1) Use the rules for logarithms to simplify the expression. 2) Use the representation of the logarithm function $u \mapsto \ln(u)$ as an integral (area under the graph of the function $u \mapsto 1/u$) to find an upper and lower bound for the given function $x \mapsto \ln\left(1+\frac{1}{x}\right)^x$ for large values of x. The bounds should be very simple functions of x.

3.3 Negative results

How can we prove that $\lim_{x \to +\infty} f(x) = L$ is false? This means that the condition (I) or the condition (II) in Definition 3.1.1 is not satisfied.

Next we formulate the negation of the condition (I): (In class I will explain how to formulate negations of statements involving "for all" and "there exists")

The negation of (I): For each $X \in \mathbb{R}$ there exists $x \geq X$ such that $f(x)$ is not defined.

Example 3.3.1. Prove that the function $f(x) = \dfrac{1}{x \operatorname{Sign}\bigl(\sin(x)\bigr)}$ does not satisfy the condition (I).

Solution. For this function the negation of (I) is true. This function is not defined for all $x = k\,\pi$ where $k \in \mathbb{Z}$. To prove that the negation of (I) is true let $X \in \mathbb{R}$ be arbitrary. Then

$$\pi \operatorname{Ceiling}\bigl(X/\pi\bigr) \geq X$$

and $f(x)$ is not defined for $x = \pi \operatorname{Ceiling}\bigl(X/\pi\bigr)$. □

Below is the plot of the function f. Small circles indicate that this function is not defined at $x = \pi, 2\pi, 3\pi, \ldots, 9\pi$.

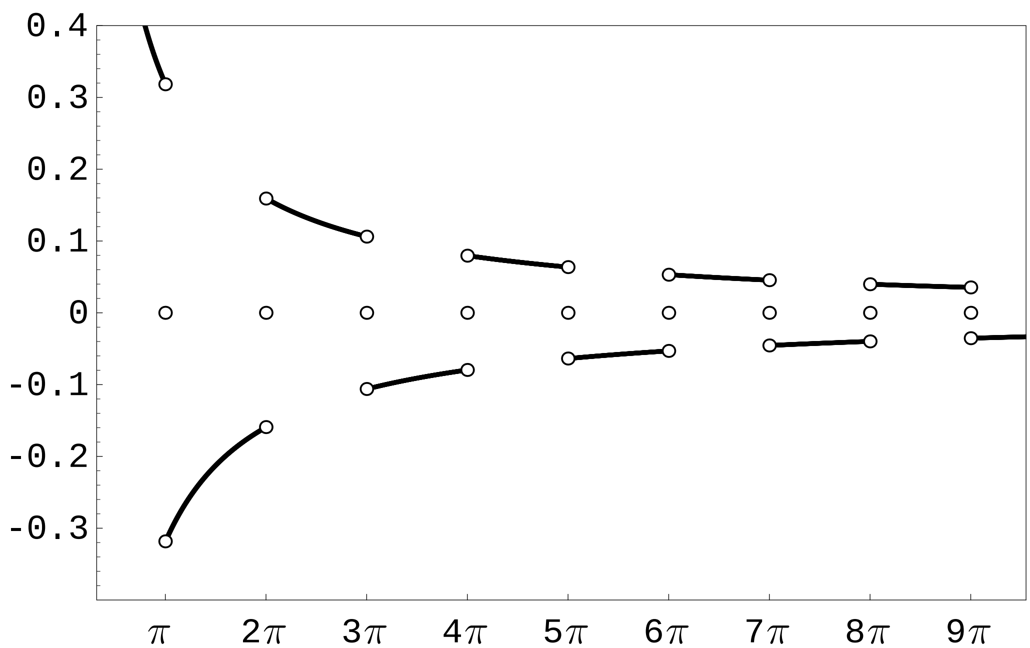


Figure 2: This function does not satisfy (I) in Definition 3.1.1

The negation of the condition (II) is more complicated:

The negation of (II): There exists $\epsilon > 0$ such that for every $X \in \mathbb{R}$ there exists $x > X$ such that $|f(x) - L| \geq \epsilon$.

Example 3.3.2. Prove that $\lim_{x\to+\infty} \sin(x) = 0$ is false.

Solution. Let $\epsilon = 1/2$. For arbitrary $X \in \mathbb{R}$ we have

$$\pi \operatorname{Ceiling}\bigl(X/\pi\bigr) + \pi/2 > X$$

and, for $x = \pi \operatorname{Ceiling}\bigl(X/\pi\bigr) + \pi/2$, we have $|\sin(x)| = 1$. Therefore

$$|\sin(x) - 0| \geq 1/2. \qquad \square$$

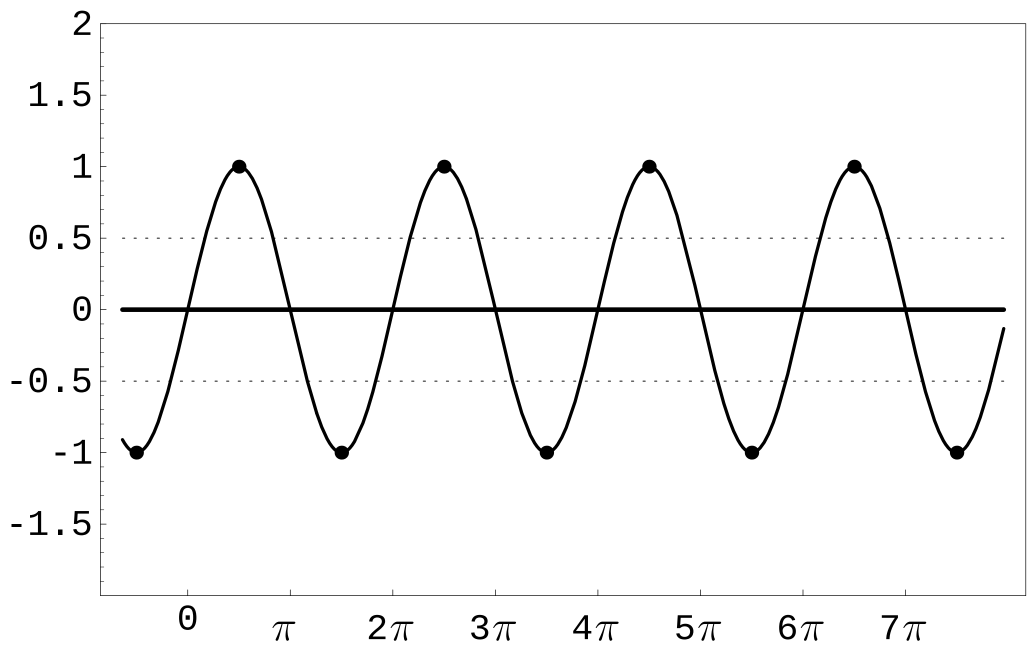


Figure 3: Illustration for the solution of Example 3.3.2

Now we consider the statement $\boxed{\text{“}\lim_{x\to+\infty} f(x) \quad \text{does not exist.”}}$

This means that for each $L \in \mathbb{R}$, $\lim_{x\to+\infty} f(x) = L$ is false.

Example 3.3.3. Prove that $\lim_{x\to+\infty} \sin(x)$ does not exist.

Solution. Let $L \in \mathbb{R}$ be arbitrary. We need to prove that $\lim_{x\to+\infty} \sin(x) = L$ is false. Consider two cases $L < 0$ and $L > 0$. Assume $L < 0$. Let $\epsilon = 1/2$. For arbitrary $X \in \mathbb{R}$ we have

$$2\pi \operatorname{Ceiling}\left(\frac{X}{2\pi}\right) + \frac{\pi}{2} > X$$

and, for $x = 2\pi \operatorname{Ceiling}\left(\frac{X}{2\pi}\right) + \frac{\pi}{2}$, we have $\sin(x) = 1$. Therefore

$$|\sin(x) - L| = |1 - L| = 1 + |L| \geq 1/2.$$

Do the case $L > 0$ as an exercise. $\square$

3.4 Infinite limits

Definition 3.4.1. A function $x \mapsto f(x)$ has the limit $+\infty$ as x approaches $+\infty$ if the following two conditions are satisfied:

(I) There exists a real number X_0 such that $f(x)$ is defined for each $x \geq X_0$.

(II) For each real number M there exists a real number $X(M) \geq X_0$ such that

$$x > X(M) \quad \Rightarrow \quad f(x) > M.$$

In this case we write $\lim_{x \to +\infty} f(x) = +\infty$.

Definition 3.4.2. A function $x \mapsto f(x)$ has the limit $-\infty$ as x approaches $+\infty$ if the following two conditions are satisfied:

(I) There exists a real number X_0 such that $f(x)$ is defined for each $x \geq X_0$.

(II) For each real number M there exists a real number $X(M) \geq X_0$ such that

$$x > X(M) \quad \Rightarrow \quad f(x) < M.$$

3.5 Examples of infinite limits

Example 3.5.1. Let $f(x) = \sqrt{x}$. Prove that $\lim_{x \to +\infty} \sqrt{x} = +\infty$.

Solution. The function $\sqrt{\cdot}$ is defined for all $x \geq 0$. Therefore we can take $X_0 = 0$ in the part (I) of the definition.

Now consider the part (II) of the definition. Let $M \in \mathbb{R}$ be arbitrary. we have to determine a real number $X(M)$ such that

$$x > X(M) \quad \Rightarrow \quad \sqrt{x} > M.$$

This will be accomplished if we solve the inequality $\sqrt{x} > M$. If $M < 0$, then all $x \geq 0$ satisfy this inequality. If $M \geq 0$ then the solution of the inequality is $x > M^2$. Thus, we can take

$$X(M) = \begin{cases} M^2 & \text{if} \quad M \geq 0, \\ 0 & \text{if} \quad M < 0\,. \end{cases}$$

Clearly, $X(M) \geq 0$ for all $M \in \mathbb{R}$ and

$$x > X(M) \quad \Rightarrow \quad \sqrt{x} > M. \qquad \square$$

Example 3.5.2. Let $f(x) = \text{Floor}(x)$. Prove that $\lim_{x \to +\infty} \text{Floor}(x) = +\infty$.

Solution. The function Floor is defined for all $x \in \mathbb{R}$. Therefore we can take $X_0 = 0$ in the part (I) of the definition.

Now consider the part (II) of the definition. Let $M \in \mathbb{R}$ be arbitrary. We have to determine a real number $X(M) \geq X_0$ such that

$$x > X(M) \quad \Rightarrow \quad \text{Floor}(x) > M. \tag{3.5.1}$$

This will be accomplished if we solve the inequality

$$\text{Floor}(x) > M. \tag{3.5.2}$$

Since we don't know much about Floor it is not easy to solve (3.5.2). To achieve the implication (3.5.1), we can replace Floor(x) in (3.5.2) with a smaller quantity $g(x)$ such that $g(x) > M$ is easy to solve. Thus we need $g(x)$ such that

(A) Floor$(x) \geq g(x)$ for all $x > X_0$.

(B) $g(x) > M$ is easy to solve.

By the definition of Floor(x) we conclude that $0 \leq x - \text{Floor}(x) < 1$ for all $x \in \mathbb{R}$. Therefore

$$x - 1 < \text{Floor}(x) \quad \text{for all} \quad x \in \mathbb{R}. \tag{3.5.3}$$

Clearly $x - 1 > M$ is easy to solve: $x > M + 1$. Thus, we can take $X(M) = \max\{M + 1, 0\}$ in the part (II) of the definition. Clearly $X(M) \geq X_0 = 0$. Let $x > X(M)$. Then $x > M + 1$ and therefore $x - 1 > M$. By the inequality (3.5.3) we conclude that

$$\text{Floor}(x) > x - 1 > M.$$

Thus $x > X(M)$ implies Floor$(x) > M$. $\square$

The key step in the solution of Example 3.5.2 was the discovery of the function $g(x)$ such that

(A) $f(x) \geq g(x)$ for all $x > X_0$.

(B) $g(x) > M$ is easy to solve.

Most proofs about limits follow this same pattern. I will sometimes refer to a discovery of the function g as a *Big Inequality.*

Exercise 3.5.3. Determine whether the following functions have the limit $+\infty$ when x approaches $+\infty$.

(a) $x \mapsto \dfrac{x^2}{2x+1}$ (b) $x \mapsto \ln x$ (c) $x \mapsto x - \sqrt{x}$

(d) $x \mapsto x - \ln(x)$ (e) $x \mapsto \dfrac{x^2 - x - 1}{x + 2\sqrt{x} + 1}$ (f) $x \mapsto \dfrac{1}{\sin\left(\frac{1}{x}\right)}$

(g) $x \mapsto \sqrt{x - \sqrt{x - \sqrt{x}}}$ (h) $x \mapsto \dfrac{(\cos x)^2 x}{\sqrt{x} + \sin(x)}$ (j) $x \mapsto \dfrac{(2 + \cos(x))x}{\sqrt{x} + \sin(x)}$

4 Limit of a function at a real number a

4.1 The definition

Definition 4.1.1. A function f has the limit $L \in \mathbb{R}$ as x approaches a real number a if the following two conditions are satisfied:

(I) There exists a real number $\delta_0 > 0$ such that $f(x)$ is defined for each x in the set $\bigl(a-\delta_0, a\bigr) \cup \bigl(a, a+\delta_0\bigr)$.

(II) For each real number $\epsilon > 0$ there exists a real number $\delta(\epsilon)$ such that $0 < \delta(\epsilon) \leq \delta_0$ and

$$0 < |x-a| < \delta(\epsilon) \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Remark 4.1.2. Notice that the condition that x belongs to the set $\bigl(a-\delta_0, a\bigr) \cup \bigl(a, a+\delta\bigr)$ can be expressed in terms of the distance between x and a as: $0 < |x-a| < \delta_0$.

The following figure illustrates Definition 4.1.1.

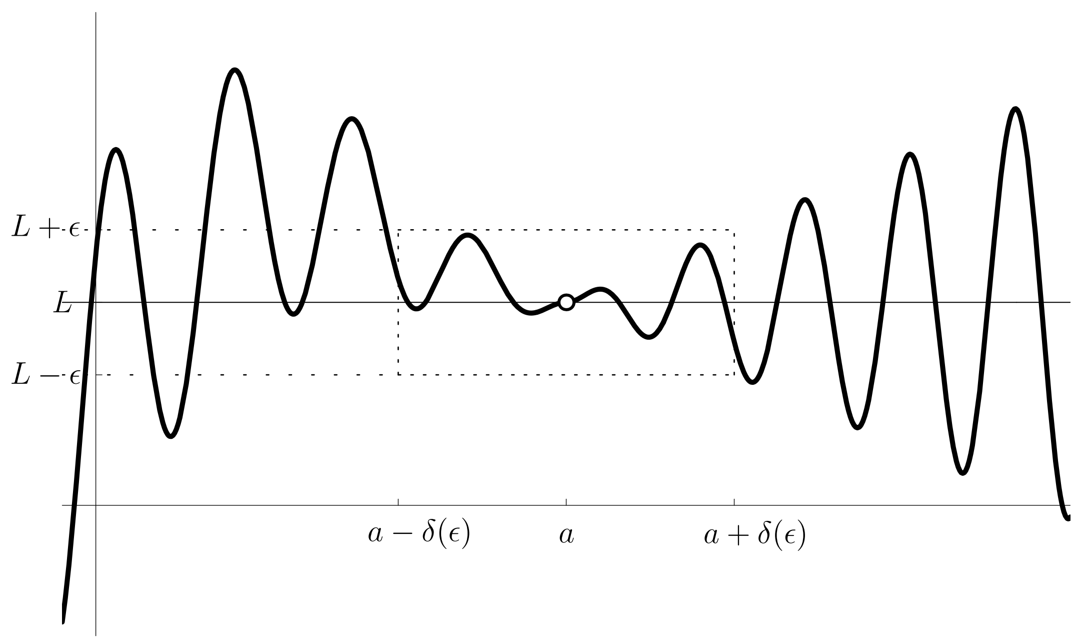
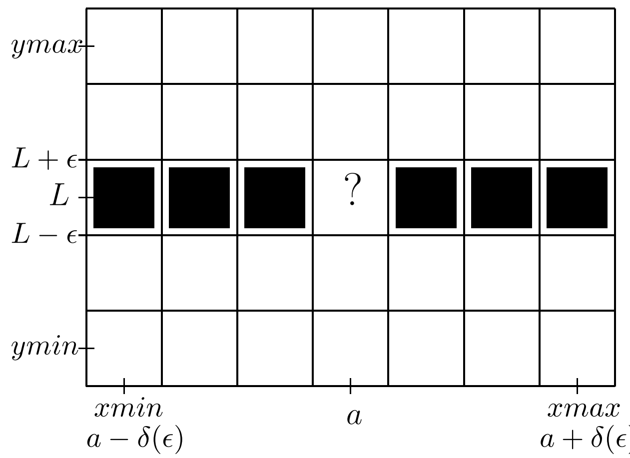


Figure 4

Next we restate Definition 4.1.1 using the terminology of a calculator screen. The figure below shows a fictional calculator screen with 35 pixels. We assume that *ymin* and *ymax* are chosen in such a way that the number L is in the middle of the y-range and that *xmin* and *xmax* are such that a is in the middle of the x-range.

Definition 4.1.3 (Calculator Screen)**.** A function f has a limit L as x approaches a if (I) in Definition 4.1.1 is satisfied and

- for each choice of $ymin$ and $ymax$ there exists Δ (which depends on $ymin$ and $ymax$) such that $0 < \Delta \leq \delta_0$ and such that whenever we choose $xmin$ and $xmax$ such that $xmax - xmin < 2\Delta$ the graph of the function f will appear to be a straight horizontal line on the calculator screen with the only possible exception at the pixel containing $x = a$.



For the specific fictional calculator screen shown above, the connection between Definition 4.1.1 and Definition 4.1.3 is given by $\epsilon = (ymax - ymin)/8$, $xmin = a - \delta(\epsilon)$, $xmax = a + \delta(\epsilon)$ and $\delta(\epsilon) = \Delta$.

The fictional screen in the example below is chosen for its simplicity. The screen of TI-92 (see the manual p. 321) is 239 pixels wide and 103 pixels tall; it has 24617 pixels. The screen of TI-83 (see the manual p. 8-16) and of TI-82 is 95 pixels wide and 63 pixels tall; it has 5985 pixels. The screen of TI-85 (see the manual p. 4-13) is 127 pixels wide and 63 pixels tall; it has 8001 pixels. The screen of TI-89 (see the manual p. 222) is 159 pixels wide and 77 pixels tall; it has 12243 pixels. Using these numbers you can calculate the connection between ϵ and $\delta(\epsilon)$ in Definition 4.1.1 and the screen of your calculator.

4.2 Examples for Definition 4.1.1

Example 4.2.1. Prove $\lim_{x \to 2} (3x - 1) = 5$.

Solution. (I) Here $f(x) = 3x - 1$. This function is defined on $\mathbb{R}$. We can take any positive number for δ_0. Since it might be useful to have a specific δ_0 to work with, we set $\delta_0 = 1$.

Let $\epsilon > 0$ be given. Let $\delta(\epsilon) = \min\{\epsilon/3, 1\}$. Assume $0 < |x - 2| < \delta(\epsilon)$. Since $\delta(\epsilon) \leq \epsilon/3$, we conclude that $|x - 2| < \epsilon/3$. Next, we calculate

$$|(3x - 1) - 5| = |3x - 6| = 3\,|x - 2|\,. \tag{4.2.1}$$

It follows from the assumption $0 < |x - 2| < \delta(\epsilon)$ that $|x - 2| < \epsilon/3$. Therefore we conclude

$$|(3x - 1) - 5| = 3\,|x - 2| < 3\frac{\epsilon}{3} = \epsilon.$$

Thus we proved that

$$0 < |x - 2| < \delta(\epsilon) \qquad \Rightarrow \qquad |(3x - 1) - 5| < \epsilon.$$

This is exactly the implication in (II) in Definition 4.1.1. Since $\epsilon > 0$ was arbitrary this completes the proof. $\square$

Remark 4.2.2. How did we guess the formula for $\delta(\epsilon)$ in the previous proof? We first studied the implication in the statement (II) in Definition 4.1.1. The goal in that implication is to prove

$$|(3x-1)-5| < \epsilon.$$

To prove this inequality we need to assume something about $|x-2|$. To find out what to assume, we simplified the expression $|(3x-1)-5|$ until $|x-2|$ appeared (see (4.2.1)). Then we solved for $|x-2|$. In this process of simplification we can afford to make the right-hand side larger. This will be illustrated in the next example.

Example 4.2.3. Prove $\lim\limits_{x\to 2}(3x^2-2x-1) = 7$.

Solution. As usual, we first deal with (I). Again $f(x) = 3x^2-2x-1$ is defined on $\mathbb{R}$ and we can take any positive number for δ_0. Since it might be useful to have a specific choice of δ_0, we put $\delta_0 = 1$. (Notice that this implies that, from now on, we consider only in the values of x which are in the set $(1,2)\cup(2,3)$.)

Next we shall discover an inequality which will help us find a formula for $\delta(\epsilon)$:

$$|(3x^2-2x-1)-7| = |3x^2-2x-8| = |(3x+4)(x-2)| = |3x+4|\,|x-2|.$$

Now we use the fact that we are considering only the values of x which are in the set $(1,2)\cup(2,3)$. For $x\in(1,2)\cup(2,3)$ the value of $|3x+4|$ does not exceed 13. Therefore

$$|(3x^2-2x-1)-7| \leq 13\,|x-2| \quad \text{for all} \quad x\in(1,2)\cup(2,3).$$

Let $\epsilon>0$ be given. The inequality $13\,|x-2|<\epsilon$ is easy to solve for $|x-2|$. The solution is $|x-2|<\epsilon/13$. Now we define $\delta(\epsilon)$:

$$\delta(\epsilon) = \min\left\{\frac{\epsilon}{13}, 1\right\}.$$

The remaining step of the proof is to prove the implication

$$|x-2|<\delta(\epsilon) \quad \Rightarrow \quad |(3x^2-2x-1)-7|<\epsilon.$$

We hope that at this point you can prove this implication on your own. □

Example 4.2.4. Prove $\lim\limits_{x\to 2}\dfrac{x^3-x-4}{x-1} = 2$.

Solution. We first deal with (I). Notice that the function $f(x) = \dfrac{x^3-x-4}{x-1}$ is defined on $\mathbb{R}\setminus\{1\}$. In this proof we are interested in the values of x near $a=2$. Therefore, for δ_0 we can take any positive number which is smaller than 1. Since it is useful to have a specific number, we put $\delta_0 = 1/2$. (Notice that this implies that from now on we consider only the values of x which are in the set $(3/2,2)\cup(2,5/2)$.)

Next we shall discover an inequality which will help us find a formula for $\delta(\epsilon)$:

$$\left|\frac{x^3-x-4}{x-1}-2\right| = \left|\frac{x^3-3x-2}{x-1}\right| = \left|\frac{(x^2+2x+1)(x-2)}{x-1}\right| = \left|\frac{x^2+2x+1}{x-1}\right|\,|x-2|. \tag{4.2.2}$$

Now remember that we are interested only in the values of x which are in the set $(3/2,2)\cup(2,5/2)$. For $x \in (3/2,2)\cup(2,5/2)$ we estimate

$$\left|\frac{x^2+2x+1}{x-1}\right| = \frac{x^2+2x+1}{x-1} \leq \frac{16}{1/2} = 32 \quad \text{for all} \quad x \in (3/2,2)\cup(2,5/2). \tag{4.2.3}$$

Combining (4.2.2) and (4.2.3) we get

$$\left|\frac{x^3-x-4}{x-1} - 2\right| \leq 32\,|x-2| \quad \text{for all} \quad x \in (3/2,2)\cup(2,5/2).$$

Let $\epsilon > 0$ be given. The inequality $32\,|x-2| < \epsilon$ is very easy to solve for $|x-2|$. The solution is $|x-2| < \epsilon/32$. Now we define $\delta(\epsilon)$:

$$\delta(\epsilon) = \min\left\{\frac{\epsilon}{32}, \frac{1}{2}\right\}.$$

The remaining piece of the proof is to prove the implication

$$|x-2| < \delta(\epsilon) \qquad \Rightarrow \qquad \left|\frac{x^3-x-4}{x-1} - 2\right| < \epsilon.$$

We hope that at this point you can prove this on your own. Write down all the details of your reasoning. □

Example 4.2.5. Prove $\lim\limits_{x\to 4} \sqrt{x} = 2$.

Solution. As usual, we first deal with (I). Notice that the function $f(x) = \sqrt{x}$ is defined on $(0,+\infty)$. We are interested in the values of x near the point $a = 4$. Thus, for δ_0 we can take any positive number which is < 4. Since it is useful to have a specific number, we put $\delta_0 = 1$. (Notice that this implies that from now on in this proof we are interested only in the values of x which are in the set $(3,4)\cup(4,5)$.)

Next we shall discover an inequality which will help us find a formula for $\delta(\epsilon)$:

$$\left|\sqrt{x}-2\right| = \left|\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}+2}\right| = \left|\frac{x-4}{\sqrt{x}+2}\right| = \left|\frac{1}{\sqrt{x}+2}\right|\,|x-4|. \tag{4.2.4}$$

Now remember that we are interested only in the values of x which are in the set $(3,4)\cup(4,5)$. For $x \in (3,4)\cup(4,5)$ we estimate

$$\left|\frac{1}{\sqrt{x}+2}\right| = \frac{1}{\sqrt{x}+2} \leq \frac{1}{\sqrt{3}+2} \leq \frac{1}{2} \quad \text{for all} \quad x \in (3,4)\cup(4,5). \tag{4.2.5}$$

Combining (4.2.4) and (4.2.5) we get

$$\left|\sqrt{x}-2\right| \leq \frac{1}{2}\,|x-4| \quad \text{for all} \quad x \in (3,4)\cup(4,5).$$

Let $\epsilon > 0$ be given. The inequality $\frac{1}{2}\,|x-4| < \epsilon$ is easy to solve for $|x-4|$. The solution is $|x-4| < 2\epsilon$. Now define $\delta(\epsilon)$:

$$\delta(\epsilon) = \min\{2\epsilon, 1\}.$$

The remaining step of the proof is to prove the implication

$$|x-4| < \min\{2\epsilon, 1\} \qquad \Rightarrow \qquad \left|\sqrt{x}-2\right| < \epsilon.$$

We hope that at this point you can prove this on your own. As before, please do it and write down the details of your reasoning. ☐

Example 4.2.6. Prove that for any $a > 0$, $\lim\limits_{x\to a} \dfrac{1}{x} = \dfrac{1}{a}$.

Solution. Let $a > 0$. As before, we first deal with (I) in Definition 4.1.1. Notice that the function $f(x) = 1/x$ is defined on $\mathbb{R}\setminus\{0\}$. We are interested in the values of x near the point $a > 0$. Thus, for δ_0 we can take any positive number which is $< a$. Since it is useful to have a specific number, we put $\delta_0 = a/2$. (Notice that this implies that from now on in this proof we are interested only in the values of x which are in the set $(a/2, a) \cup (a, 3a/2)$.)

Next we shall discover an inequality which will help us find a formula for $\delta(\epsilon)$:

$$\left|\frac{1}{x} - \frac{1}{a}\right| = \left|\frac{a-x}{xa}\right| = \frac{|a-x|}{xa} = \frac{1}{xa}\,|x-a|\,. \tag{4.2.6}$$

Now remember that we are interested only in the values of x which are in the set $(a/2, a) \cup (a, 3a/2)$. For $x \in (a/2, a) \cup (a, 3a/2)$ we estimate

$$\frac{1}{xa} \le \frac{1}{(a/2)a} = \frac{2}{a^2} \quad \text{for all} \quad x \in (a/2, a) \cup (a, 3a/2). \tag{4.2.7}$$

Combining (4.2.6) and (4.2.7) we get

$$\left|\frac{1}{x} - \frac{1}{a}\right| \le \frac{2}{a^2}\,|x-a| \quad \text{for all} \quad x \in (a/2, a) \cup (a, 3a/2).$$

Let $\epsilon > 0$ be given. The inequality $\frac{2}{a^2}\,|x-a| < \epsilon$ is easy to solve for $|x-a|$. The solution is $|x-a| < (a^2/2)\epsilon$. Now define $\delta(\epsilon)$:

$$\delta(\epsilon) = \min\left\{\frac{a^2}{2}\,\epsilon, \frac{a}{2}\right\}.$$

The remaining step of the proof is to prove the implication

$$|x-a| < \min\left\{\frac{a^2}{2}\,\epsilon, \frac{a}{2}\right\} \qquad \Rightarrow \qquad \left|\frac{1}{x} - \frac{1}{a}\right| < \epsilon.$$

We hope that at this point you can prove this on your own. Write down the details of your reasoning. ☐

Exercise 4.2.7. Find each of the following limits. Prove your claims using Definition 4.1.1.

(a) $\lim_{x\to 3}(2x+1)$ (b) $\lim_{x\to 1}(-3x-7)$ (c) $\lim_{x\to 1}(4x^2+3)$

(d) $\lim_{x\to 2}\frac{x}{x-1}$ (e) $\lim_{x\to 3}\frac{x^2-x+2}{x+1}$ (f) $\lim_{x\to 0}x^{1/3}$

(g) $\lim_{x\to 0}\left(\frac{1}{|x|}\right)^{3/\ln|x|}$ (h) $\lim_{x\to 0}\tan x$ (i) $\lim_{x\to 0}\frac{1}{\cos x}$

(j) $\lim_{x\to 3}\frac{1}{x}$ (k) $\lim_{x\to 1}\frac{1}{x^2+1}$ (l) $\lim_{x\to -2}\frac{x}{x^2+4\,x+3}$

Exercise 4.2.8. Let $f(x)=\dfrac{x+1}{x^2-1}$. Does f have a limit at $a=1$? Justify your answer.

Exercise 4.2.9. Prove that for any $a>0$, $\lim_{x\to a}\sqrt{x}=\sqrt{a}$.

4.3 Infinite limits

Definition 4.3.1. A function f has the limit $+\infty$ as x approaches a real number a if the following two conditions are satisfied:

(I) There exists a real number $\delta_0>0$ such that $f(x)$ is defined for each x in the set $\big(a-\delta_0,a\big)\cup\big(a,a+\delta_0\big)$.

(II) For each real number $M>0$ there exists a real number $\delta(M)$ such that $0<\delta(M)\le\delta_0$ and

$$0<|x-a|<\delta(\epsilon)\quad\Rightarrow\quad f(x)>M.$$

Definition 4.3.2. A function f has the limit $-\infty$ as x approaches a real number a if the following two conditions are satisfied:

(I) There exists a real number $\delta_0>0$ such that $f(x)$ is defined for each x in the set $\big(a-\delta_0,a\big)\cup\big(a,a+\delta_0\big)$.

(II) For each real number $M<0$ there exists a real number $\delta(M)$ such that $0<\delta(M)\le\delta_0$ and

$$0<|x-a|<\delta(\epsilon)\quad\Rightarrow\quad f(x)<M.$$

Exercise 4.3.3. Find each of the following limits. Prove your claims using the appropriate definition.

(a) $\lim_{x\to 0}\frac{1}{|x|}$ (b) $\lim_{x\to -3}\frac{1}{(x+3)^2}$ (c) $\lim_{x\to 2}\frac{x-3}{x(x-2)^2}$

(d) $\lim_{x\to -1}\frac{x}{(x+1)^4}$ (e) $\lim_{x\to +\infty}\frac{x^2-x+2}{x+1}$ (f) $\lim_{x\to +\infty}\frac{x^2-x}{3-x}$

4.4 One-sided limits

Definition 4.4.1. A function f has the limit $L \in \mathbb{R}$ as x approaches a real number a from the left if the following two conditions are satisfied:

(I) There exists a real number $\delta_0 > 0$ such that $f(x)$ is defined for each x in the set $\bigl(a-\delta_0, a\bigr)$.

(II) For each real number $\epsilon > 0$ there exists a real number $\delta(\epsilon)$ such that $0 < \delta(\epsilon) \leq \delta_0$ and

$$0 < a - x < \delta(\epsilon) \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

If the conditions (I) and (II) in Definition 4.4.1 are satisfied we write $\lim\limits_{x \uparrow a} f(x) = L$.

Definition 4.4.2. A function f has the limit $L \in \mathbb{R}$ as x approaches a real number a from the right if the following two conditions are satisfied:

(I) There exists a real number $\delta_0 > 0$ such that $f(x)$ is defined for each x in the set $\bigl(a, a+\delta_0\bigr)$.

(II) For each real number $\epsilon > 0$ there exists a real number $\delta(\epsilon)$ such that $0 < \delta(\epsilon) \leq \delta_0$ and

$$0 < x - a < \delta(\epsilon) \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

If the conditions (I) and (II) in Definition 4.4.2 are satisfied we write $\lim\limits_{x \downarrow a} f(x) = L$.

Definition 4.4.3. A function f has the limit $+\infty$ as x approaches a real number a from the left if the following two conditions are satisfied:

(I) There exists a real number $\delta_0 > 0$ such that $f(x)$ is defined for each x in the set $\bigl(a-\delta_0, a\bigr)$.

(II) For each real number $M > 0$ there exists a real number $\delta(M)$ such that $0 < \delta(M) \leq \delta_0$ and

$$0 < a - x < \delta(M) \quad \Rightarrow \quad f(x) > M.$$

If the conditions (I) and (II) in Definition 4.4.3 are satisfied we write $\lim\limits_{x \uparrow a} f(x) = +\infty$.

Definition 4.4.4. A function f has the limit $+\infty$ as x approaches a real number a from the right if the following two conditions are satisfied:

(I) There exists a real number $\delta_0 > 0$ such that $f(x)$ is defined for each x in the set $\bigl(a, a+\delta_0\bigr)$.

(II) For each real number $M > 0$ there exists a real number $\delta(M)$ such that $0 < \delta(M) \leq \delta_0$ and

$$0 < x - a < \delta(M) \quad \Rightarrow \quad f(x) > M.$$

If the conditions (I) and (II) in Definition 4.4.4 are satisfied we write $\lim\limits_{x \downarrow a} f(x) = +\infty$.

Definition 4.4.5. A function f has the limit $-\infty$ as x approaches a real number a from the left if the following two conditions are satisfied:

(I) There exists a real number $\delta_0 > 0$ such that $f(x)$ is defined for each x in the set $\big(a-\delta_0, a\big)$.

(II) For each real number $M < 0$ there exists a real number $\delta(M)$ such that $0 < \delta(M) \leq \delta_0$ and
$$0 < a - x < \delta(M) \quad \Rightarrow \quad f(x) < M.$$

If the conditions (I) and (II) in Definition 4.4.5 are satisfied we write $\lim\limits_{x \uparrow a} f(x) = -\infty$.

Definition 4.4.6. A function f has the limit $-\infty$ as x approaches a real number a from the right if the following two conditions are satisfied:

(I) There exists a real number $\delta_0 > 0$ such that $f(x)$ is defined for each x in the set $\big(a, a+\delta_0\big)$.

(II) For each real number $M < 0$ there exists a real number $\delta(M)$ such that $0 < \delta(M) \leq \delta_0$ and
$$0 < x - a < \delta(M) \quad \Rightarrow \quad f(x) < M.$$

If the conditions (I) and (II) in Definition 4.4.6 are satisfied we write $\lim\limits_{x \downarrow a} f(x) = -\infty$.

Exercise 4.4.7. Find each of the following limits. Prove your claims using the appropriate definition.

(a) $\displaystyle \lim_{x \uparrow 5} \frac{3x - 15}{\sqrt{x^2 - 10x + 25}}$ (b) $\displaystyle \lim_{x \downarrow 5} \frac{3x - 15}{\sqrt{x^2 - 10x + 25}}$ (c) $\displaystyle \lim_{x \uparrow 2} \frac{x - 3}{x(x-2)}$

(d) $\displaystyle \lim_{x \downarrow 0} \left(\frac{1}{x} - \frac{1}{x^2}\right)$ (e) $\displaystyle \lim_{x \uparrow 5} \frac{2}{\sqrt{5 - x}}$ (f) $\displaystyle \lim_{x \downarrow 5} \frac{6}{5 - x}$

(g) $\displaystyle \lim_{x \uparrow 3} \frac{x + 3}{x^2 - 9}$ (h) $\displaystyle \lim_{x \uparrow -3} \frac{x^2}{x^2 - 9}$ (i) $\displaystyle \lim_{x \downarrow 0} \big(x - \sqrt{x}\big)$

(j) $\displaystyle \lim_{x \to 3} \frac{x}{(x-3)^2}$ (k) $\displaystyle \lim_{x \downarrow -1} \frac{x^2}{x + 1}$ (l) $\displaystyle \lim_{x \to +\infty} \big(x - \sqrt{x}\big)$

5 New limits from old

5.1 Squeeze theorems

In this section and in Section 5.3 we establish general properties of limits which are based on the formal definition of limit. These properties are stated as theorems.

Establishing theorems of this kind involves a major step forward in sophistication. Up to this point we have been trying to show that limits exist directly from the definition. Now for the first time we are going to **assume** that some limit exists (I refer to this in class as a *green* limit.) and try to make use of this information to establish the existence of some other limit (I refer to this in class as a *red* limit.). Remember that to establish the existence of a limit, we had to come up with a procedure for finding $\delta(\epsilon)$ that will work for any $\epsilon > 0$ that is given. If we assume the existence of a limit, then we are assuming the existence of such a procedure, though we may not know explicitly what it is. I refer to this as a *green* $\delta(\epsilon)$. It is this procedure we will need to use in order to construct a new procedure for the limit whose existence we are trying to establish. I refer to this as a *red* $\delta(\epsilon)$.

We start by considering squeeze theorems that resemble the role of BIN in previous sections. The following theorem is the Sandwich Squeeze Theorem.

Theorem 5.1.1. *Let f, g and h be given functions and let a and L be real numbers. Suppose that the following three conditions are satisfied.*

(1) $\lim\limits_{x\to a} f(x) = L$,

(2) $\lim\limits_{x\to a} h(x) = L$,

(3) *There exists $\eta_0 > 0$ such that $f(x), g(x)$ and $h(x)$ are defined for all $x \in \big(a-\eta_0, a\big)\cup\big(a, a+\eta_0\big)$ and*

$$f(x) \le g(x) \le h(x) \quad \textit{for all} \quad x \in \big(a-\eta_0, a\big)\cup\big(a, a+\eta_0\big).$$

Then

$$\lim_{x\to a} g(x) = L.$$

Proof. Here we have three functions and three definitions of limits, one for each function. Therefore we have to deal with three δ-s. We shall give them appropriate names that will distinguish them from each other. Let us name them δ_f, δ_g and δ_h.

In the theorem it is assumed that $\lim\limits_{x\to a} f(x) = L$. This means that we are given the fact that for each $\epsilon > 0$ there exists $\delta_f(\epsilon) > 0$ (that is, we are given a function $\delta_f(\epsilon)$) such that

$$0 < |x-a| < \delta_f(\epsilon) \quad \Rightarrow \quad |f(x) - L| < \epsilon. \tag{5.1.1}$$

In class I refer to these as a green $\delta_f(\cdot)$ and a green implication.

Since the theorem assumes that $\lim\limits_{x\to a} h(x) = L$, we are also given that for each $\epsilon > 0$ there exists $\delta_h(\epsilon) > 0$ such that

$$0 < |x-a| < \delta_h(\epsilon) \quad \Rightarrow \quad |h(x) - L| < \epsilon. \tag{5.1.2}$$

Again we refer to these as a green $\delta_h(\cdot)$ and a green implication.

We need to prove that $\lim_{x\to a} g(x) = L$. Therefore, following the definition of limit, we have to show that the following conditions are satisfied:

(I) There exists a real number $\delta_{0,g} > 0$ such that $g(x)$ is defined for each x in the set $\big(a - \delta_{0,g}, a\big) \cup \big(a, a + \delta_{0,g}\big)$.

(II) For each real number $\epsilon > 0$ there exists a real number $\delta_g(\epsilon)$ such that $0 < \delta_g(\epsilon) \leq \delta_{0,g}$ and such that

$$0 < |x - a| < \delta_g(\epsilon) \quad \Rightarrow \quad |g(x) - L| < \epsilon. \tag{5.1.3}$$

Since we have to produce $\delta_{0,g}, \delta_g(\epsilon)$ and we have to prove the last implication, all of these objects are red.

Notice that η_0 in the theorem is green.

The objective here is to use the green objects to produce the red objects. We shall do that next. We put:

(I) $\delta_{0,g} = \eta_0$. By the assumption of the theorem $g(x)$ is defined for each x in the set $\big(a - \eta_0, a\big) \cup \big(a, a + \eta_0\big)$.

(II) For each real number $\epsilon > 0$, put

$$\delta_g(\epsilon) = \min\big\{\delta_f(\epsilon), \delta_h(\epsilon), \eta_0\big\}.$$

This is a beautiful expression since the red object is expressed in terms of the green objects.

It remains to prove the red implication (5.1.3) using the green implications and the assumptions of the theorem.

To prove (5.1.3), $\boxed{\text{assume that } 0 < |x - a| < \delta_g(\epsilon).}$ Then, clearly, $0 < |x - a| < \eta_0$. This is telling me that $x \neq a$ and that x is no further than η_0 from a. Consequently, $x \in \big(a - \eta_0, a\big) \cup \big(a, a + \eta_0\big)$. Therefore, by the assumption of the theorem

$$f(x) \leq g(x) \leq h(x).$$

Subtracting L from each term in this inequality, we conclude that

$$f(x) - L \leq g(x) - L \leq h(x) - L.$$

Using the property of the absolute value that $-|u| \leq u \leq |u|$ for each real number u, we conclude that

$$-|f(x) - L| \leq f(x) - L \leq g(x) - L \leq h(x) - L \leq |h(x) - L|. \tag{5.1.4}$$

From the assumption $0 < |x - a| < \delta_g(\epsilon)$, we conclude that $0 < |x - a| < \delta_f(\epsilon)$. By the green implication (5.1.1), this implies that $|f(x) - L| < \epsilon$ and therefore

$$-\epsilon < -|f(x) - L|. \tag{5.1.5}$$

From the assumption $0 < |x - a| < \delta_g(\epsilon)$, we conclude that $0 < |x - a| < \delta_h(\epsilon)$. By the green implication (5.1.2), this implies that

$$|h(x) - L| < \epsilon. \tag{5.1.6}$$

Putting together the inequalities (5.1.4), (5.1.5) and (5.1.6), we conclude that

$$-\epsilon < g(x) - L < \epsilon. \tag{5.1.7}$$

The inequalities in (5.1.7) are equivalent to

$$|g(x) - L| < \epsilon.$$

This proves that $0 < |x-a| < \delta_g(\epsilon)$ implies $|g(x)-L| < \epsilon$ and this is exactly the red implication (5.1.3). This completes the proof. $\square$

The following theorem is the Scissors Squeeze Theorem.

Theorem 5.1.2. *Let f, g and h be given functions and let $a \in \mathbb{R}$ and $L \in \mathbb{R}$. Assume that*

(1) $\displaystyle\lim_{x\to a} f(x) = L$,

(2) $\displaystyle\lim_{x\to a} h(x) = L$,

(3) *There exists $\eta_0 > 0$ such that $f(x), g(x)$ and $h(x)$ are defined for all $x \in \bigl(a-\eta_0, a\bigr)\cup\bigl(a, a+\eta_0\bigr)$ and*

$$f(x) \leq g(x) \leq h(x) \quad \text{for all} \quad x \in \bigl(a-\eta_0, a\bigr),$$

and

$$h(x) \leq g(x) \leq f(x) \quad \text{for all} \quad x \in \bigl(a, a+\eta_0\bigr).$$

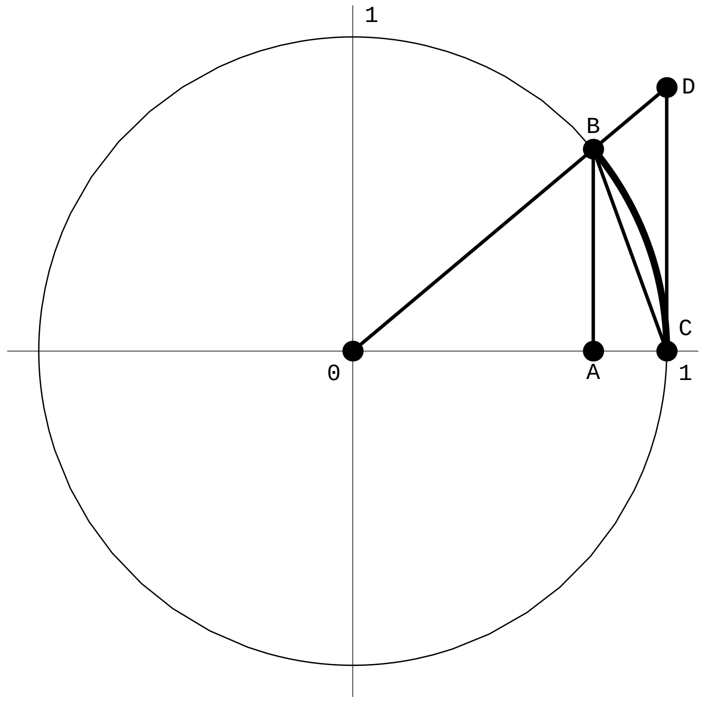
Then

$$\lim_{x\to a} g(x) = L.$$

5.2 Examples for squeeze theorems

The following picture and the numbers that *you can see on it* are essential for getting squeezes for limits involving trigonometric functions. The table to the left shows the numbers that you should be able to identify on the picture.

Geometric Object	Associated Number
Circular Arc from C to B	u
Line Segment $\overline{0A}$	$\cos u$
Line Segment $\overline{AB}$	$\sin u$
Line Segment $\overline{AC}$	$1 - \cos u$
Line Segment $\overline{CB}$	You Calculate
Line Segment $\overline{CD}$	$\tan u$
Line Segment $\overline{0B}$	1
Line Segment $\overline{0C}$	1



Example 5.2.1. Prove that $\lim_{x\to 0} \cos x = 1$.

Solution. Set $\eta_0 = \dfrac{\pi}{3}$. Consider positive u. Look at the picture above. The triangle $\triangle ACB$ is a right triangle. Therefore its hypothenuse, the line segment $\overline{CB}$, is longer than its side $\overline{AC}$ which equals to $1 - \cos u$. Thus

$$1 - \cos u = \overline{AC} \leq \overline{CB}. \tag{5.2.1}$$

The line segment $\overline{CB}$ is a segment of a straight line, therefore it is shorter than any other curve joining C and B. In particular it is shorter than the circular arc joining the points C and B. The length of this circular arc is u. Thus

$$\overline{CB} \leq \quad \text{Length of the Circular Arc from } C \text{ to } B \ (= \ u\ ). \tag{5.2.2}$$

Putting together the inequalities (5.2.1) and (5.2.2), we conclude that

$$1 - \cos u \leq u \quad \text{ for all } \quad 0 < u < \frac{\pi}{3}. \tag{5.2.3}$$

Since the length $\overline{0A} = \cos u$ is smaller than 1, from (5.2.3) we conclude that

$$0 \leq 1 - \cos u \leq u \quad \text{ for all } \quad 0 < u < \frac{\pi}{3}\ ,$$

or, equivalently,

$$1 - u \leq \cos u \leq 1 \quad \text{ for all } \quad 0 < u < \frac{\pi}{3}\ ,$$

Now we substitute $u = |x|$ and use the fact that $\cos|x| = \cos x$ and (5.2) becomes

$$1 - |x| \leq \cos x \leq 1 \quad \text{ for all } \quad -\frac{\pi}{3} < x < \frac{\pi}{3}.$$

This is a sandwich squeeze for $\cos x$. It is easy to prove that $\lim_{x\to 0} 1 = 1$ and $\lim_{x\to 0}\big(1 - |x|\big) = 1$. (Please prove this using the definition!) Now the Sandwich Squeeze Theorem implies that $\lim_{x\to 0} \cos x = 1$. $\square$

Example 5.2.2. Prove that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$.

Solution. To get a sandwich squeeze for this problem consider the following three areas on the picture above.

Area 1 The triangle $\triangle 0CB$.

Area 2 The segment of the unit disc bounded by the line segments $\overline{0C}$ and $\overline{0B}$ and the circular arc segment joining points C and B.

Area 3 The triangle $\triangle 0CD$.

The picture tells clearly the inequality between these areas. Write that inequality. Calculate each area in terms of the numbers that appear in the table above. This will lead to the inequality, which when simplified gives

$$\cos u \leq \frac{\sin u}{u} \leq 1 \quad \text{for all} \quad 0 < x < \frac{\pi}{3}. \tag{5.2.4}$$

Using the same idea as in the previous example, the inequality (5.2.4) leads to

$$\cos x \leq \frac{\sin x}{x} \leq 1 \quad \text{for all} \quad x \in \left(-\frac{\pi}{3}, 0\right) \cup \left(0, \frac{\pi}{3}\right). \tag{5.2.5}$$

The inequality (5.2.5) is exactly what we need in the Sandwich Squeeze Theorem. Please fill in all the details of the rest of the proof. $\square$

Example 5.2.3. Prove that $\lim_{x\to 0} \dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$.

Solution. To establish squeeze inequlaities consider three lengths:

Length 1 The line segment $\overline{AB}$.

Length 2 The line segment $\overline{CB}$.

Length 3 The length of a circular arc joining the points C and B.

The picture tells clearly the inequalities between these three lengths. Write these inequalities. Calculate each length in terms of the numbers that appear in the table above. This will lead to the inequalities, which, when simplified, give

$$\frac{1}{2}\left(\frac{\sin u}{u}\right)^2 \leq \frac{1-\cos u}{u^2} \leq \frac{1}{2} \quad \text{for all} \quad 0 < u < \frac{\pi}{3}. \tag{5.2.6}$$

From the inequality (5.2.6) and one inequality established in a previous example you can get an "easy" sandwich squeeze. Please fill in all the details of the rest of the proof. $\square$

Example 5.2.4. Prove that $\lim_{x\to 0} \dfrac{\ln(1+x)}{x} = 1$.

Solution. The idea is to use the definition of $\ln$ as an integral and work with areas to get squeeze inequalities. $\square$

5.3 Algebra of limits

A nickname that I gave to a function which has a limit L when x approaches a is: f is *constantish* L near a. If we are dealing with constant functions $f(x) = L$ and $g(x) = K$, then clearly the sum $f + g$ of these two functions is a constant function equal to $L + K$. The same is true for the product fg which is the constant function equal to LK. Another question is whether we can talk about the reciprocal $1/f$. If $L \neq 0$, then the reciprocal of f is defined and it equals $1/L$. In this section we shall prove that all these properties hold for constantish functions.

Theorem 5.3.1. *Let f, g, and h, be functions with domain and range in $\mathbb{R}$. Let a, K and L be real numbers. Assume that*

(1) $\lim\limits_{x\to a} f(x) = K$,

(2) $\lim\limits_{x\to a} g(x) = L$.

Then the following statements hold.

(A) *If $h = f + g$, then $\lim\limits_{x\to a} h(x) = K + L$.*

(B) *If $h = fg$, then $\lim\limits_{x\to a} h(x) = KL$.*

(C) *If $L \neq 0$ and $h = \dfrac{1}{g}$, then $\lim\limits_{x\to a} h(x) = \dfrac{1}{L}$.*

(D) *If $L \neq 0$ and $h = \dfrac{f}{g}$, then $\lim\limits_{x\to a} h(x) = \dfrac{K}{L}$.*

Proof. The assumption $\lim\limits_{x\to a} f(x) = K$ implies that

green(I-f) There exists (green!) $\delta_{0,f} > 0$ such that $f(x)$ is defined for all x in $\bigl(a - \delta_{0,f}, a\bigr) \cup \bigl(a, a + \delta_{0,f}\bigr)$;

green(II-f) For each $\epsilon > 0$ there exists (green!) $\delta_f(\epsilon)$ such that $0 < \delta_f(\epsilon) \leq \delta_{0,f}$ and such that

$$0 < |x - a| < \delta_f(\epsilon) \quad \Rightarrow \quad |f(x) - K| < \epsilon. \tag{5.3.1}$$

The assumption $\lim\limits_{x\to a} g(x) = L$ implies that

green(I-g) There exists (green!) $\delta_{0,g} > 0$ such that $g(x)$ is defined for all x in $\bigl(a - \delta_{0,g}, a\bigr) \cup \bigl(a, a + \delta_{0,g}\bigr)$;

green(II-g) For each $\epsilon > 0$ there exists (green!) $\delta_g(\epsilon)$ such that $0 < \delta_g(\epsilon) \leq \delta_{0,g}$ and such that

$$0 < |x - a| < \delta_g(\epsilon) \quad \Rightarrow \quad |g(x) - L| < \epsilon. \tag{5.3.2}$$

Proof of the statement (A). Remember that $h(x) = f(x) + g(x)$ here. First we list what is red in this proof.

red(I-h) There exists (red!) $\delta_{0,h} > 0$ such that $h(x)$ is defined for all x in $\big(a-\delta_{0,h}, a\big) \cup \big(a, a+\delta_{0,h}\big)$;

red(II-h) For each $\epsilon > 0$ there exists (red!) $\delta_h(\epsilon)$ such that $0 < \delta_h(\epsilon) \leq \delta_{0,h}$ and such that

$$0 < |x-a| < \delta_h(\epsilon) \quad \Rightarrow \quad |h(x) - (K+L)| < \epsilon. \tag{5.3.3}$$

I will not elaborate here how I got the idea for $\delta_{0,h}$ and $\delta_h(\epsilon)$, I will just give formulas and convince you that my choice is a correct one. The idea for the formulas comes from the boxed paragraph on page 28. I invite you to enjoy the separation of colors in the following formulas.

Let $\epsilon > 0$ be given. Put

$$\delta_{0,h} := \min\{\delta_{0,f}, \delta_{0,g}\}$$
$$\delta_h(\epsilon) := \min\left\{\delta_f\left(\frac{\epsilon}{2}\right), \delta_g\left(\frac{\epsilon}{2}\right)\right\}$$

Now we have to prove that $h(x)$ is defined for each $x \in \big(a-\delta_{0,h}, a\big) \cup \big(a, a+\delta_{0,h}\big)$. Assume that $x \in \big(a-\delta_{0,h}, a\big) \cup \big(a, a+\delta_{0,h}\big)$. Then

$$0 < |x-a| < \delta_{0,h} \leq \min\{\delta_{0,f}, \delta_{0,g}\}. \tag{5.3.4}$$

It follows from (5.3.4) that

$$0 < |x-a| < \delta_{0,f},$$

and therefore $x \in \big(a-\delta_{0,f}, a\big) \cup \big(a, a+\delta_{0,f}\big)$. Thus $f(x)$ is defined. It also follows from (5.3.4) that

$$0 < |x-a| < \delta_{0,g},$$

and therefore $x \in \big(a-\delta_{0,g}, a\big) \cup \big(a, a+\delta_{0,g}\big)$. Thus $g(x)$ is defined. Therefore $h(x) = f(x) + g(x)$ is defined for each $x \in \big(a-\delta_{0,h}, a\big) \cup \big(a, a+\delta_{0,h}\big)$.

Now we will prove the red implication (5.3.3). Assume

$$0 < |x-a| < \delta_h(\epsilon) = \min\left\{\delta_f\left(\frac{\epsilon}{2}\right), \delta_g\left(\frac{\epsilon}{2}\right)\right\}. \tag{5.3.5}$$

Then

$$0 < |x-a| < \delta_f\left(\frac{\epsilon}{2}\right). \tag{5.3.6}$$

The inequality (5.3.6) and the implication (5.3.1) allow me to conclude that

$$|f(x) - K| < \frac{\epsilon}{2}. \tag{5.3.7}$$

It follows from (5.3.5) that

$$0 < |x-a| < \delta_g\left(\frac{\epsilon}{2}\right). \tag{5.3.8}$$

The inequality (5.3.8) and the implication (5.3.2) allow me to conclude that

$$|g(x) - L| < \frac{\epsilon}{2}. \tag{5.3.9}$$

Now we remember that the absolute value has the property that $|u+v| \leq |u|+|v|$. We will apply this to the expression

$$|h(x)-(K+L)| = |f(x)+g(x)-K-L| = |\underbrace{(f(x)-K)}_{u} + \underbrace{(g(x)-L)}_{v}|$$

to get

$$|h(x)-(K+L)| \leq |f(x)-K| + |g(x)-L|. \tag{5.3.10}$$

This inequality plays a role of a BIN in this abstract proof. It has an unfriendly object on the left and all friendly objects on the right.

The inequalities (5.3.7), (5.3.9) and (5.3.10) imply that

$$|h(x)-(K+L)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{5.3.11}$$

Reviewing my reasoning above you should be convinced that based on the assumption (5.3.5) we proved the inequality (5.3.11). This is exactly the implication (5.3.3). This completes the proof of the statement (A).

Proof of the statement (B). Remember that $h(x) = f(x)g(x)$ here. We first list what is red in this proof.

red(I-h) There exists (red!) $\delta_{0,h} > 0$ such that $h(x)$ is defined for all x in $\big(a-\delta_{0,h}, a\big) \cup \big(a, a+\delta_{0,h}\big)$;

red(II-h) For each $\epsilon > 0$ there exists (red!) $\delta_h(\epsilon)$ such that $0 < \delta_h(\epsilon) \leq \delta_{0,h}$ and such that

$$0 < |x-a| < \delta_h(\epsilon) \quad \Rightarrow \quad |h(x)-KL| < \epsilon. \tag{5.3.12}$$

I will not elaborate how I got the idea for $\delta_{0,h}$ and $\delta_h(\epsilon)$, I will just give formulas and convince you that my choice is a correct one. The idea for the formulas comes from the boxed paragraph on page 30. Again, I invite you to enjoy the separation of colors in the following formulas.

Let $\epsilon > 0$ be given. Put

$$\begin{aligned}\delta_{0,h} &:= \min\left\{\delta_{0,f}, \delta_g(1)\right\}\\ \delta_h(\epsilon) &:= \min\left\{\delta_f\left(\frac{\epsilon}{2(|L|+1)}\right), \delta_g\left(\frac{\epsilon}{2(|K|+1)}\right)\right\}\end{aligned}$$

Now we have to prove that $h(x)$ is defined for each $x \in \big(a-\delta_{0,h}, a\big) \cup \big(a, a+\delta_{0,h}\big)$. Assume that $x \in \big(a-\delta_{0,h}, a\big) \cup \big(a, a+\delta_{0,h}\big)$. Then

$$0 < |x-a| < \delta_{0,h} \leq \min\left\{\delta_{0,f}, \delta_g(1)\right\}. \tag{5.3.13}$$

It follows from (5.3.13) that

$$0 < |x-a| < \delta_{0,f}\ ,$$

and therefore $x \in \big(a - \delta_{0,f}, a\big) \cup \big(a, a + \delta_{0,f}\big)$. Thus $f(x)$ is defined. It also follows from (5.3.13) that

$$0 < |x - a| < \delta_g(1). \tag{5.3.14}$$

Since by the assumption (II-g) we know that $\delta_g(1) \leq \delta_{0,g}$, the inequality (5.3.14) implies that

$$0 < |x - a| < \delta_{0,g}.$$

Therefore $x \in \big(a - \delta_{0,g}, a\big) \cup \big(a, a + \delta_{0,g}\big)$. Thus $g(x)$ is defined. Therefore $h(x) = f(x)g(x)$ is defined for each $x \in \big(a - \delta_{0,h}, a\big) \cup \big(a, a + \delta_{0,h}\big)$.

At this point we will prove another consequence of the inequality (5.3.14). This inequality and the implication (5.3.2) allow me to conclude that

$$|g(x) - L| < 1.$$

Therefore

$$-1 < g(x) - L < 1 \ ,$$

or, equivalently

$$-1 + L < g(x) < L + 1.$$

Multiplying the last inequality by -1, we conclude that

$$-1 - L < -g(x) < -L + 1.$$

From the last two inequalities we conclude that $\max\{g(x), -g(x)\} < \max\{L + 1, -L + 1\} = \max\{L, -L\} + 1$. Thus

$$|g(x)| < |L| + 1. \tag{5.3.15}$$

Now we will prove the red implication (5.3.12). Assume

$$0 < |x - a| < \delta_h(\epsilon) = \min\left\{\delta_f\left(\frac{\epsilon}{2(|L| + 1)}\right), \delta_g\left(\frac{\epsilon}{2(|K| + 1)}\right)\right\}. \tag{5.3.16}$$

Then

$$0 < |x - a| < \delta_f\left(\frac{\epsilon}{2(|K| + 1)}\right). \tag{5.3.17}$$

The inequality (5.3.17) and the implication (5.3.1) allow me to conclude that

$$|f(x) - K| < \frac{\epsilon}{2(|L| + 1)}. \tag{5.3.18}$$

It follows from (5.3.16) that

$$0 < |x - a| < \delta_g\left(\frac{\epsilon}{2(|K| + 1)}\right). \tag{5.3.19}$$

The inequality (5.3.19) and the implication (5.3.2) allow me to conclude that

$$|g(x) - L| < \frac{\epsilon}{2(|K| + 1)}. \tag{5.3.20}$$

Now we remember that the absolute value has the property that $|u+v| \leq |u| + |v|$ and that $|uv| = |u||v|$. we will apply these properties to the expression

$$\begin{aligned}|h(x) - KL| = |f(x)g(x) - KL| &= |\underbrace{\big(f(x)g(x) - Kg(x)\big)}_{u} + \underbrace{\big(Kg(x) - KL\big)}_{v}| \\ &\leq |f(x)g(x) - Kg(x)\big)| + |Kg(x) - KL| \\ &\leq |g(x)|\,|f(x) - K| + |K|\,|g(x) - L|.\end{aligned}$$

Summarizing

$$|h(x) - KL| \leq |g(x)|\,|f(x) - K| + |K|\,|g(x) - L|. \tag{5.3.21}$$

The inequalities (5.3.15) and (5.3.21) imply that

$$|h(x) - KL| \leq \big(|L| + 1\big)\,|f(x) - K| + |K|\,|g(x) - L|. \tag{5.3.22}$$

This inequality plays a role of a BIN in this abstract proof. It has an unfriendly object on the left and all friendly objects on the right.

The inequalities (5.3.18), (5.3.20) and (5.3.22) imply that

$$|h(x) - LK| \leq \big(|L| + 1\big)\,\frac{\epsilon}{2(|L|+1)} + |K|\,\frac{\epsilon}{2(|K|+1)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{5.3.23}$$

I hope that my reasoning above convinces you that the assumption (5.3.16) implies the inequality (5.3.23). This is exactly the implication (5.3.12). This completes the proof of the part (B).

Proof of the statement (C). Here we assume that $L \neq 0$ and $h(x) = \dfrac{1}{g(x)}$. Next we list what is red in this proof.

red(I-h) There exists (red!) $\delta_{0,h} > 0$ such that $h(x)$ is defined for all x in $\big(a - \delta_{0,h}, a\big) \cup \big(a, a + \delta_{0,h}\big)$;

red(II-h) For each $\epsilon > 0$ there exists (red!) $\delta_h(\epsilon)$ such that $0 < \delta_h(\epsilon) \leq \delta_{0,h}$ and such that

$$0 < |x - a| < \delta_h(\epsilon) \quad \Rightarrow \quad \left|\frac{1}{g(x)} - \frac{1}{L}\right| < \epsilon. \tag{5.3.24}$$

I will not elaborate how I got the idea for $\delta_{0,h}$ and $\delta_h(\epsilon)$, I will just give formulas and convince you that my choice is a correct one. The idea for the formulas comes from the boxed paragraph on page 32. Again, I invite you to enjoy the separation of colors in the following formulas.

Let $\epsilon > 0$ be given. Remember that it is assumed that $|L| > 0$. Put

$$\begin{aligned}\delta_{0,h} &:= \delta_g\left(\frac{|L|}{2}\right) \\ \delta_h(\epsilon) &:= \min\left\{\delta_g\left(\frac{\epsilon L^2}{2}\right), \delta_g\left(\frac{|L|}{2}\right)\right\}.\end{aligned}$$

Now we have to prove that $h(x)$ is defined for each $x \in \big(a-\delta_{0,h}, a\big) \cup \big(a, a+\delta_{0,h}\big)$. Assume that $x \in \big(a-\delta_{0,h}, a\big) \cup \big(a, a+\delta_{0,h}\big)$. Then

$$0 < |x-a| < \delta_{0,h} = \delta_g\left(\frac{|L|}{2}\right).$$

This inequality and the implication (5.3.2) allow me to conclude that

$$|g(x) - L| < \frac{|L|}{2}.$$

Therefore

$$-\frac{|L|}{2} < g(x) - L < \frac{|L|}{2}\ ,$$

or, equivalently

$$-\frac{|L|}{2} + L < g(x) < L + \frac{|L|}{2}.$$

Multiplying the last inequality by -1, we conclude that

$$-L - \frac{|L|}{2} < -g(x) < \frac{|L|}{2} - L.$$

From the last two displayed relationships we conclude that

$$\max\{g(x), -g(x)\} > \max\left\{L - \frac{|L|}{2}, -L - \frac{|L|}{2}\right\} = \max\{L, -L\} - \frac{|L|}{2}.$$

Thus

$$|g(x)| > |L| - \frac{|L|}{2} = \frac{|L|}{2} > 0. \tag{5.3.25}$$

Consequently, $g(x) \neq 0$. Therefore, $h(x) = \dfrac{1}{g(x)}$ is defined for all $x \in \big(a-\delta_{0,h}, a\big) \cup \big(a, a+\delta_{0,h}\big)$.

Now we will prove the red implication (5.3.24). Assume

$$0 < |x-a| < \delta_h(\epsilon) = \min\left\{\delta_g\left(\frac{\epsilon L^2}{2}\right), \delta_g\left(\frac{|L|}{2}\right)\right\}. \tag{5.3.26}$$

Then

$$0 < |x-a| < \delta_g\left(\frac{\epsilon L^2}{2}\right). \tag{5.3.27}$$

The inequality (5.3.27) and the implication (5.3.2) allow me to conclude that

$$|g(x) - L| < \frac{\epsilon L^2}{2}. \tag{5.3.28}$$

It also follows from (5.3.26) that

$$0 < |x-a| < \delta_g\left(\frac{|L|}{2}\right).$$

We already proved that this inequality implies (5.3.25). Therefore

$$\frac{1}{|g(x)|} < \frac{2}{|L|}. \tag{5.3.29}$$

This inequality is used at the last step in the sequence of inequalities below. In some sense this is an abstract version of a "pizza-party" play.

Using our standard tools, algebra, properties of the absolute value and the inequality (5.3.29) we get

$$\begin{aligned}\left|h(x) - \frac{1}{L}\right| &= \left|\frac{1}{g(x)} - \frac{1}{L}\right| = \left|\frac{L-g(x)}{g(x)L}\right| = \frac{|L-g(x)|}{|g(x)|\ |L|}\\ &= \frac{|g(x)-L|}{|g(x)|\ |L|} \leq \frac{1}{|g(x)|}\ \frac{|g(x)-L|}{|L|} \leq \frac{2}{|L|}\ \frac{|g(x)-L|}{|L|}.\end{aligned}$$

Summarizing

$$\left|\frac{1}{g(x)} - \frac{1}{L}\right| \leq \frac{2}{L^2}\ |g(x) - L|\,. \tag{5.3.30}$$

This inequality plays a role of a BIN in this abstract proof. It has an unfriendly object on the left and all friendly objects on the right.

The inequalities (5.3.28) and (5.3.30) imply that

$$\left|\frac{1}{g(x)} - \frac{1}{L}\right| \leq \frac{2}{L^2}\ \frac{\epsilon L^2}{2} = \epsilon. \tag{5.3.31}$$

I hope that the reasoning above convinces you that the assumption (5.3.26) implies the inequality (5.3.31). This is exactly the implication (5.3.24). This completes the proof of the part (C).

Proof of the statement (D). Here we assume that $L \neq 0$ and $h(x) = \dfrac{f(x)}{g(x)}$. We can prove the statement (D) by using the universal power of the statements (B) and (C). First define the functions $g_1(x) = \dfrac{1}{g(x)}$. Then, by the statement (C) we know

$$\lim_{x\to a} g_1(x) = \frac{1}{L}. \tag{5.3.32}$$

Clearly, $h(x) = f(x)g_1(x)$. Now we can apply the statement (B) to this function h. Taking into account (5.3.32) the statement (B) implies

$$\lim_{x\to a} h(x) = K\,\frac{1}{L} = \frac{K}{L}.$$

This completes the proof of the statement (D). The theorem is proved. □

Exercise 5.3.2. Use the algebra of limits to give much simpler proofs for most of the limits in the previous exercises and examples.

5.4 L'Hospital rule

By definition $f'(a) = \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$.

Theorem 5.4.1. *Let f and g be functions and let a be a real number such that $f(a) = g(a) = 0$. Assume that the derivatives $f'(a)$ and $g'(a)$ exist and $g'(a) \neq 0$. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Proof. Assume that the limits $f'(a) = \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ and $g'(a) = \lim_{x\to a} \dfrac{g(x)-g(a)}{x-a}$ exist and $g'(a) \neq 0$. Then the limit

$$\lim_{x\to a} \frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}} \tag{5.4.1}$$

exists and it equals $\dfrac{f'(a)}{g'(a)}$. Remember that $f(a) = g(a) = 0$ and simplify

$$\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}} = \frac{\dfrac{f(x)}{x-a}}{\dfrac{g(x)}{x-a}} = \frac{f(x)}{g(x)}. \tag{5.4.2}$$

Based on (5.4.1) and (5.4.2) we conclude that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

□

The following is a more powerful version of the L'Hospital rule. It's proof is not that much more complicated, but we will skip it here.

Theorem 5.4.2. *Let f and g be functions and let a be a real number such that $f(a) = g(a) = 0$. Assume that there exists $\delta_0 > 0$ such that $f(x), g(x), f'(x), g'(x)$ are defined for all $x \in \big(a-\delta_0, a\big) \cup \big(a, a+\delta_0\big)$. Assume that*

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L.$$

Then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = L.$

Example 5.4.3. Calculate $\lim_{x\to 0} \dfrac{x-\sin x}{x^3}$.

Solution. Put $f(x) = x - \sin x$ and $g(x) = x^3$. Put $\delta_0 = 1$. Then $f(x)$ and $g(x)$ is defined for all $x \in (-1, 1)$. Let $x \in (-1, 1)$ and calculate $f'(x) = 1 - \cos x$ and $g'(x) = 3x^2$. Now calculate

$$\begin{aligned}\lim_{x\to 0} \frac{f'(x)}{g'(x)} &= \lim_{x\to 0} \frac{1-\cos x}{3x^2} = \lim_{x\to 0} \frac{1}{3} \cdot \frac{1-\cos x}{x^2} \\ &= \frac{1}{3} \cdot \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}\end{aligned}$$

□

Exercise 5.4.4. Use the L'Hospital Rule to find each of the following limits.

(a) $\displaystyle\lim_{x\to 1} \frac{x^9 - 1}{x^5 - 1}$ (b) $\displaystyle\lim_{x\to 1} \frac{x^a - 1}{x^b - 1}$ (c) $\displaystyle\lim_{x\to \pi/2} \frac{1 - \sin x}{\cos x}$

(d) $\displaystyle\lim_{x\to 1} \frac{\ln x}{x - 1}$ (e) $\displaystyle\lim_{x\to 0} \frac{1 - \cos x}{(\sin x)^2}$ (f) $\displaystyle\lim_{x\to 0} \frac{\ln(1 + x)}{\sin x}$

(g) $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{x}$ (h) $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$ (i) $\displaystyle\lim_{x\to 0} \frac{x + \tan x}{\sin x}$

6 Continuous functions

6.1 Definition and examples

All this work about limits will now pay off since we shall be able to give mathematically rigorous definition of a continuous function.

Definition 6.1.1. Let f be a real valued function of a real variable and let a be a real number. The function f is continuous at a if the following two conditions are satisfied:

(i) The function f is defined at a, that is $f(a)$ is defined.

(ii) $\displaystyle \lim_{x\to a} f(x) = f(a)$.

To understand Definition 6.1.1 the reader has to understand the concept of limit. Sometimes it is useful to state the definition of continuity directly, without appealing to the concept of limit.

Definition 6.1.2. Let f be a real valued function of a real variable and let a be a real number. The function f is continuous at a if the following two conditions are satisfied:

(I) There exists a $\delta_0 > 0$ such that $f(x)$ is defined for all $x \in (a-\delta_0, a+\delta_0)$.

(II) For each $\epsilon > 0$ there exists $\delta(\epsilon)$ such that $0 < \delta(\epsilon) \leq \delta_0$ and such that

$$|x-a| < \delta(\epsilon) \quad \Rightarrow \quad |f(x) - f(a)| < \epsilon.$$

Definition 6.1.2 is called ϵ-δ definition of continuity.

Definition 6.1.3. Let I be an interval in $\mathbb{R}$. A function f is *continuous on* I if it is continuous at each point in I.

Example 6.1.4. Let c be a real number and define $f(x) = c$ for all $x \in \mathbb{R}$. Use Definition 6.1.2 to prove that f is continuous at an arbitrary real number a.

Example 6.1.5. Let $f(x) = x$ for all $x \in \mathbb{R}$. Use Definition 6.1.2 to prove that f is continuous at an arbitrary real number a.

Example 6.1.6. Use ϵ-δ definition of continuity, that is Definition 6.1.2, to prove that the function $f(x) = 1/x$ is continuous on the interval $(0, +\infty)$.

Solution. Let $a \in (0, +\infty)$, that is let a be an arbitrary positive number. Chose $\delta_0 = a/2$. Since $a > 0$, we conclude that $a/2 > 0$ and $f(x) = 1/x$ is defined for all $x \in \bigl(a/2, 3a/2\bigr)$.

Let $\epsilon > 0$ be arbitrary. Now we have to solve

$$\left| \frac{1}{x} - \frac{1}{a} \right| < \epsilon \qquad \text{for} \qquad |x-a|.$$

First simplify the expression, using the fact that $x > 0$ and $a > 0$ and rules for the absolute value:

$$\left| \frac{1}{x} - \frac{1}{a} \right| = \left| \frac{a-x}{x\,a} \right| = \frac{|a-x|}{|x|\,|a|} = \frac{|x-a|}{x\,a}.$$

To get a larger expression which will be easy to solve we replace x in the denominator by the smallest possible value for x. That value is $a - a/2 = a/2$. This gives me my BIN:

$$\left|\frac{1}{x} - \frac{1}{a}\right| = \frac{|x-a|}{x\,a} \leq \frac{|x-a|}{\dfrac{a}{2}\,a} = 2\,\frac{|x-a|}{a^2}.$$

Thus my BIN is $\displaystyle \left|\frac{1}{x} - \frac{1}{a}\right| \leq 2\,\frac{|x-a|}{a^2}$ valid for all $x \in \big(a/2, 3a/2\big)$.

Solving the inequality $\displaystyle 2\,\frac{|x-a|}{a^2} < \epsilon$ for $|x-a|$ is easy. The solution is $|x-a| < a^2\,\epsilon/2$. Now we define

$$\delta(\epsilon) = \min\left\{\frac{a^2\,\epsilon}{2}, \frac{a}{2}\right\}.$$

To finish the proof, it remains to prove the implication

$$|x-a| < \delta(\epsilon) \quad \Rightarrow \quad \left|\frac{1}{x} - \frac{1}{a}\right| < \epsilon.$$

This should be easy, using the BIN. $\square$

Example 6.1.7. Use ϵ-δ definition of continuity, that is Definition 6.1.2, to prove that the function $x \mapsto \sqrt{x}$ is continuous on the interval $(0, +\infty)$.

Solution. Let $a \in (0, +\infty)$. Chose $\delta_0 = \dfrac{a}{2}$. Since $a > 0$, as before we conclude that $\dfrac{a}{2} > 0$ and the function $x \mapsto \sqrt{x}$ is defined for all $x \in (a/2, 3a/2)$.

Let $\epsilon > 0$ be arbitrary. Now we have to solve

$$\big|\sqrt{x} - \sqrt{a}\big| < \epsilon \quad \text{for} \quad |x-a|.$$

First simplify algebraically the expression, using the fact that $x > 0$ and $a > 0$ and rules for the absolute value.

$$\begin{aligned} \big|\sqrt{x} - \sqrt{a}\big| &= \left|\big(\sqrt{x} - \sqrt{a}\big)\,\frac{1}{1}\right| = \left|\big(\sqrt{x} - \sqrt{a}\big)\,\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}\right| = \left|\frac{x-a}{\sqrt{x}+\sqrt{a}}\right| \\ &= \frac{|x-a|}{\big|\sqrt{x}+\sqrt{a}\big|} = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} \leq \frac{|x-a|}{\sqrt{a}} \end{aligned}$$

Thus my BIN is: $\displaystyle \big|\sqrt{x} - \sqrt{a}\big| \leq \frac{|x-a|}{\sqrt{a}}$, valid for $x > 0$.

Solving $\dfrac{|x-a|}{\sqrt{a}} < \epsilon$ for $|x-a|$ is easy: The solution is $|x-a| < \sqrt{a}\,\epsilon$. Now we define

$$\delta(\epsilon) = \min\left\{\sqrt{a}\,\epsilon, \frac{a}{2}\right\}.$$

It remains to prove the implication $|x-a| < \min\left\{\sqrt{a}\,\epsilon, \dfrac{a}{2}\right\} \quad \Rightarrow \quad \big|\sqrt{x} - \sqrt{a}\big| < \epsilon$. This should be easy, using the BIN. $\square$

Example 6.1.8. Let $f(x) = \dfrac{1}{x^2+1}$ for all $x \in \mathbb{R}$. Use ϵ-δ definition to prove that f is continuous at an arbitrary $a \in \mathbb{R}$.

The following theorem follows from Theorem 5.3.1.

Theorem 6.1.9 (Algebra of Continuous Functions)**.** *Let f and g be functions and let a be a real number. Assume that f and g are continuous at the point a.*

(a) *If $h = f + g$, then h is continuous at a.*

(b) *If $h = fg$, then h is continuous at a.*

(c) *If $h = \dfrac{f}{g}$ and $g(a) \neq 0$, then h is continuous at a.*

Example 6.1.10. Let a, b, c be any real numbers. Let $f(x) = ax^2 + bx + c$ for all $x \in \mathbb{R}$. Let v be an arbitrary real number. Prove that f is continuous at v.

Example 6.1.11. Let $f(x) = \dfrac{1}{x^2+1}$ for all $x \in \mathbb{R}$. Use algebra of continuous functions to prove that f is continuous on $\mathbb{R}$.

Example 6.1.12. Let $f(x) = \sin x$ for all $x \in \mathbb{R}$. Prove that f is continuous at an arbitrary real number a.

Solution. The proof consists of two steps:

(1) Use the definition of limit and inequalities that we proved for $\sin$ and $\cos$ to prove the following two limits

$$\lim_{x\to a} \sin(x-a) = 0\,,$$
$$\lim_{x\to a} \cos(x-a) = 1\,.$$

(2) Use the addition formula for $\sin x = \sin(x - a + a)$ and the algebra of limits to complete the proof. □

Example 6.1.13. Let $f(x) = \cos x$ for all $x \in \mathbb{R}$. Prove that f is continuous at an arbitrary real number a.

Example 6.1.14. Let $f(x) = \tan x$ for all $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$. Prove that f is continuous at an arbitrary real number a such that $-\dfrac{\pi}{2} < a < \dfrac{\pi}{2}$.

Solution. Use the algebra of continuous functions. □

Example 6.1.15. Let $f(x) = \ln x$ for all $x \in (0, +\infty)$. Prove that f is continuous at an arbitrary real number a.

Solution. Use the definition of ln to derive the squeeze for ln:

$$1 - \frac{1}{x} \leq \ln x \leq x - 1, \qquad 0 < x < +\infty.$$

Use the above squeeze to prove that for arbitrary $a > 0$ we have $\lim_{x \to a} \ln\left(\frac{x}{a}\right) = 0$. Now use the rule for logarithms $\ln uv = \ln u + \ln v$. $\square$

Example 6.1.16. Let $f(x) = e^x$ for all $x \in \mathbb{R}$. Prove that f is continuous at an arbitrary real number a.

Solution. Use the fact that $x \mapsto e^x$ is the inverse of the logarithm function to derive the squeeze for it:

$$1 + x \leq e^x \leq \frac{1}{1-x}, \qquad -\infty < x < 1.$$

Get the rest of the proof as an exercise. $\square$

The following theorem claims that a composition of continuous functions is continuous. I will post a proof on the class website.

Theorem 6.1.17. *Let f and g be functions and let a be a real number. Assume that g is continuous at a and that f is continuous at $g(a)$. If $h = f \circ g$, then h is continuous at a.*