7 Sequences of real numbers

7.1 Definitions and examples

Definition 7.1.1. A sequence of real numbers is a real function whose domain is the set $\mathbb{N}$ of natural numbers.

Let $s : \mathbb{N} \to \mathbb{R}$ be a sequence. Then the values of s are $s(1), s(2), s(3), \ldots, s(n), \ldots$. It is customary to write s_n instead of $s(n)$ in this case. Sometimes a sequence will be specified by listing its first few terms

$$s_1,\ s_2,\ s_3,\ s_4,\ \ldots\ ,$$

and sometimes by listing of all its terms $\{s_n\}_{n\in\mathbb{N}}$ or $\{s_n\}_{n=1}^{+\infty}$. One way of specifying a sequence is to give a formula, or recursion formula for its $n-$th term s_n. Notice that in this notation s is the "name" of the sequence and n is the variable.

Some examples of sequences follow.

Example 7.1.2. (a) $1,\ 0,\ -1,\ 0,\ 1,\ 0,\ -1,\ \ldots\ ;$

(b) $1,\ 0,\ 1,\ 1,\ 0,\ 1,\ 1,\ 1,\ 0,\ 1,\ 1,\ 1,\ 1,\ 0,\ 1,\ \ \ldots\ ;$

(c) $1,\ 1,\ 1,\ 1,\ 1,\ \ldots\ ;$ (the constant sequence)

(d) $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \ldots$; (What is the range of this sequence?)

Recursively defined sequences

Example 7.1.3. (a) $x_1 = 1, \quad x_{n+1} = 1 + \dfrac{x_n}{4},\ n = 1, 2, 3, \ldots;$

(b) $x_1 = 2, \quad x_{n+1} = \dfrac{x_n}{2} + \dfrac{1}{x_n},\ n = 1, 2, 3, \ldots;$

(c) $a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2 + a_n}, \quad n = 1, 2, 3, \ldots;$

(d) $s_1 = 1, \quad s_{n+1} = \sqrt{1 + s_n}, \quad n = 1, 2, 3, \ldots;$

(e) $x_1 = \dfrac{9}{10}, \quad x_{n+1} = \dfrac{9 + x_n}{10}, \quad n = 1, 2, 3, \ldots.$

(f) $b_1 = \dfrac{1}{2}, \quad b_{n+1} = \dfrac{1}{2\sqrt{1 - b_n^2}}, \quad n = 1, 2, 3, \ldots$

(g) $f_1 = 1,\ f_{n+1} = (n + 1)\, f_n, \quad n = 1, 2, 3, \ldots.$

Some important examples of sequences are listed below.

$$b_n = c, \quad c \in \mathbb{R}. \quad n \in \mathbb{N}, \tag{7.1.1}$$
$$p_n = a^n, \quad a \in \mathbb{R}, \quad n \in \mathbb{N}, \tag{7.1.2}$$
$$x_n = \left(1 + \frac{1}{n}\right)^n, \quad n \in \mathbb{N}, \tag{7.1.3}$$
$$y_n = \left(1 + \frac{1}{n}\right)^{(n+1)}, \quad n \in \mathbb{N}, \tag{7.1.4}$$
$$z_n = \left(1 + \frac{a}{n}\right)^n, \quad a \in \mathbb{R}, \quad n \in \mathbb{N}, \tag{7.1.5}$$
$$f_1 = 1, \quad f_{n+1} = f_n \cdot (n+1), \quad n \in \mathbb{N}. \tag{7.1.6}$$

(The standard notation for the terms of the sequence $\{f_n\}_{n=1}^{+\infty}$ is $f_n = n!, \ n \in \mathbb{N}$.)

$$q_n = \frac{a^n}{n!}, \quad a \in \mathbb{R}, \quad n \in \mathbb{N}, \tag{7.1.7}$$
$$t_1 = 2, \ t_{n+1} = t_n + \frac{1}{n!} \quad n \in \mathbb{N}, \tag{7.1.8}$$
$$v_1 = 1 + a, \ v_{n+1} = v_n + \frac{a^n}{n!}, \quad a \in \mathbb{R}, \quad n \in \mathbb{N}. \tag{7.1.9}$$

Let $\{a_n\}_{n=1}^{+\infty}$ be an arbitrary sequence. An important sequence associated with $\{a_n\}_{n=1}^{+\infty}$ is the following sequence

$$S_1 = a_1, \ S_{n+1} = S_n + a_{n+1}, \quad n \in \mathbb{N}. \tag{7.1.10}$$

7.2 Convergent sequences

Definition 7.2.1. A sequence $\{s_n\}_{n=1}^{+\infty}$ of real numbers *converges* to the real number L if for each $\epsilon > 0$ there exists a number $N(\epsilon)$ such that

$$n \in \mathbb{N}, \quad n > N(\epsilon) \qquad \Rightarrow \qquad |s_n - L| < \epsilon.$$

If $\{s_n\}_{n=1}^{+\infty}$ converges to L we will write

$$\lim_{n \to +\infty} s_n = L \qquad \text{or} \qquad s_n \to L \ \ (n \to +\infty).$$

The number L is called the *limit* of the sequence $\{s_n\}_{n=1}^{+\infty}$. A sequence that does not converge to a real number is said to *diverge*.

Example 7.2.2. Let r be a real number such that $|r| < 1$. Prove that $\lim_{n \to +\infty} r^n = 0$.

Solution. First note that if $r = 0$, then $r^n = 0$ for all $n \in \mathbb{N}$, so the given sequence is a constant sequence. Therefore it converges. Let $\epsilon > 0$. We need to solve $|r^n - 0| < \epsilon$ for n. First simplify $|r^n - 0| = |r^n| = |r|^n$. Now solve $|r|^n < \epsilon$ by taking ln of both sides of the inequality (note that ln is an increasing function)

$$\ln |r|^n = n \ln |r| < \ln \epsilon.$$

Since $|r| < 1$, we conclude that $\ln|r| < 0$. Therefore the solution is $n > \dfrac{\ln\epsilon}{\ln|r|}$. Thus, with $N(\epsilon) = \dfrac{\ln\epsilon}{\ln|r|}$, the implication

$$n \in \mathbb{N}, \quad n > N(\epsilon) \quad \Rightarrow \quad |r^n - 0| < \epsilon$$

is valid. □

Example 7.2.3. Prove that $\displaystyle\lim_{n\to+\infty} \frac{n^2 - n - 1}{2n^2 - 1} = \frac{1}{2}$.

Solution. Let $\epsilon > 0$ be given. We need to solve $\left|\dfrac{n^2 - n - 1}{2n^2 - 1} - \dfrac{1}{2}\right| < \epsilon$ for n. First simplify:

$$\left|\frac{n^2 - n - 1}{2n^2 - 1} - \frac{1}{2}\right| = \left|\frac{2}{2}\,\frac{n^2 - n - 1}{2n^2 - 1} - \frac{1}{2}\,\frac{2n^2 - 1}{2n^2 - 1}\right| = \left|\frac{-2n - 1}{2\,(2n^2 - 1)}\right| = \frac{2n + 1}{4n^2 - 2}$$

Now invent the BIN:

$$\frac{2n + 1}{4n^2 - 2} \leq \frac{2n + n}{4n^2 - 2n^2} = \frac{3n}{2n^2} = \frac{3}{2n}.$$

Therefore the BIN is:

$$\left|\frac{n^2 - n - 1}{2n^2 - 1} - \frac{1}{2}\right| \leq \frac{3}{2n} \quad \text{valid for} \quad n \in \mathbb{N}.$$

Solving for n is now easy:

$$\frac{3}{2n} < \epsilon. \qquad \text{The solution is} \quad n > \frac{3}{2\epsilon}.$$

Thus, with $N(\epsilon) = \dfrac{3}{2\epsilon}$, the implication

$$n > N(\epsilon) \quad \Rightarrow \quad \left|\frac{n^2 - n - 1}{2n^2 - 1} - 1\right| < \epsilon$$

is valid. Using the BIN, this implication should be easy to prove. □

This procedure is very similar to the procedure for proving limits as x approaches infinity. In fact the following two theorems are true.

Theorem 7.2.4. *Let $x \mapsto f(x)$ be a function which is defined for every $x \geq 1$. Define the sequence $a : \mathbb{N} \to \mathbb{R}$ by*

$$a_n = f(n) \quad \textit{for every} \quad n \in \mathbb{N}.$$

If $\displaystyle\lim_{x\to+\infty} f(x) = L$, then $\displaystyle\lim_{n\to+\infty} a_n = L$.

Theorem 7.2.5. *Let $x \mapsto f(x)$ be a function which is defined for every $x \in (0, 1]$. Define the sequence $a : \mathbb{N} \to \mathbb{R}$ by*

$$a_n = f(1/n) \quad \textit{for every} \quad n \in \mathbb{N}.$$

If $\displaystyle\lim_{x\downarrow 0} f(x) = L$, then $\displaystyle\lim_{n\to+\infty} a_n = L$.

The above two theorems are useful for proving limits of sequences which are defined by a formula. For example you can prove the following limits by using these two theorems and what we proved in previous sections.

Exercise 7.2.6. Find the following limits. Provide proofs.

(a) $\displaystyle\lim_{n\to+\infty} \sin\left(\frac{1}{n}\right)$ (b) $\displaystyle\lim_{n\to+\infty} n\sin\left(\frac{1}{n}\right)$ (c) $\displaystyle\lim_{n\to+\infty} \ln\left(1+\frac{1}{n}\right)$

(d) $\displaystyle\lim_{n\to+\infty} n\ln\left(1+\frac{1}{n}\right)$ (e) $\displaystyle\lim_{n\to+\infty} \cos\left(\frac{1}{n}\right)$ (f) $\displaystyle\lim_{n\to+\infty} \frac{1}{n}\cos\left(\frac{1}{n}\right)$

The Algebra of Limits Theorem holds for sequences.

Theorem 7.2.7. *Let $\{a_n\}_{n=1}^{+\infty}$, $\{b_n\}_{n=1}^{+\infty}$ and $\{c_n\}_{n=1}^{+\infty}$, be given sequences. Let K and L be real numbers. Assume that*

(1) $\displaystyle\lim_{x\to+\infty} a_n = K,$

(2) $\displaystyle\lim_{x\to+\infty} b_n = L.$

Then the following statements hold.

(A) *If $c_n = a_n + b_n$, $n \in \mathbb{N}$, then $\displaystyle\lim_{x\to+\infty} c_n = K + L.$*

(B) *If $c_n = a_n b_n$, $n \in \mathbb{N}$, then $\displaystyle\lim_{x\to+\infty} c_n = KL.$*

(C) *If $L \neq 0$ and $c_n = \dfrac{a_n}{b_n}$, $n \in \mathbb{N}$, then $\displaystyle\lim_{x\to+\infty} c_n = \frac{K}{L}.$*

Theorem 7.2.8. *Let $\{a_n\}_{n=1}^{+\infty}$ and $\{b_n\}_{n=1}^{+\infty}$ be given sequences. Let K and L be real numbers. Assume that*

(1) $\displaystyle\lim_{x\to+\infty} a_n = K.$

(2) $\displaystyle\lim_{x\to+\infty} b_n = L.$

(3) *There exists a natural number n_0 such that*

$$a_n \leq b_n \quad \text{for all} \quad n \geq n_0.$$

Then $K \leq L$.

Theorem 7.2.9. *Let $\{a_n\}_{n=1}^{+\infty}$, $\{b_n\}_{n=1}^{+\infty}$ and $\{s_n\}_{n=1}^{+\infty}$ be given sequences. Assume the following*

1. *The sequence $\{a_n\}_{n=1}^{+\infty}$ converges to the limit L.*
2. *The sequence $\{b_n\}_{n=1}^{+\infty}$ converges to the limit L.*
3. *There exists a natural number n_0 such that*

$$a_n \leq s_n \leq b_n \quad \text{for all} \quad n > n_0.$$

Then the sequence $\{s_n\}_{n=1}^{+\infty}$ converges to the limit L.

Prove this theorem.

7.3 Sufficient conditions for convergence

Many limits of sequences cannot be found using theorems from the previous section. For example, the recursively defined sequences (a), (b), (c), (d) and (e) in Example 7.1.3 converge but it cannot be proved using the theorems that we presented so far.

Definition 7.3.1. Let $\{s_n\}_{n=1}^{+\infty}$ be a sequence of real numbers.

1. If a real number M satisfies
$$s_n \leq M \quad \text{for all} \quad n \in \mathbb{N}$$
then M is called an *upper bound* of $\{s_n\}_{n=1}^{+\infty}$ and the sequence $\{s_n\}_{n=1}^{+\infty}$ is said to be *bounded above.*

2. If a real number m satisfies
$$m \leq s_n \quad \text{for all} \quad n \in \mathbb{N},$$
then m is called a *lower bound* of $\{s_n\}_{n=1}^{+\infty}$ and the sequence $\{s_n\}_{n=1}^{+\infty}$ is said to be *bounded below.*

3. The sequence $\{s_n\}_{n=1}^{+\infty}$ is said to be *bounded* if it is bounded above and bounded below.

Theorem 7.3.2. *If a sequence converges, then it is bounded.*

Proof. Assume that a sequence $\{a_n\}_{n=1}^{+\infty}$ converges to L. By Definition 7.2.1 this means that for each $\epsilon > 0$ there exists a number $N(\epsilon)$ such that

$$n \in \mathbb{N}, \quad n > N(\epsilon) \qquad \Rightarrow \qquad |a_n - L| < \epsilon.$$

In particular for $\epsilon = 1 > 0$ there exists a number $N(1)$ such that

$$n \in \mathbb{N}, \quad n > N(1) \qquad \Rightarrow \qquad |a_n - L| < 1.$$

Let n_0 be the largest natural number which is $\leq \ N(1)$. Then $n_0 + 1, n_0 + 2, \dots$ are all $> \ N(1)$. Therefore
$$|a_n - L| < 1 \quad \text{for all} \quad n > n_0.$$
This means that
$$L - 1 < a_n < L + 1 \quad \text{for all} \quad n > n_0.$$
The numbers $L - 1$ and $L + 1$ are not lower and upper bounds for the sequence since we do not know how they relate to the first n_0 terms of the sequence. Put

$$m = \min\{a_1, a_2, \dots, a_{n_0}, L - 1\}$$
$$M = \max\{a_1, a_2, \dots, a_{n_0}, L + 1\}.$$

Clearly

$$\begin{aligned} m &\leq a_n && \text{for all} \quad n = 1, 2, \dots, n_0 \\ m \leq L - 1 &< a_n && \text{for all} \quad n > n_0. \end{aligned}$$

Thus m is a lower bound for the sequence $\{a_n\}_{n=1}^{+\infty}$.

Clearly

$$\begin{aligned} a_n &\leq M \qquad &&\text{for all} \qquad n = 1, 2, \dots, n_0 \\ a_n < L + 1 &\leq M \qquad &&\text{for all} \qquad n > n_0. \end{aligned}$$

Thus M is an upper bound for the sequence $\{a_n\}_{n=1}^{+\infty}$. □

Is the converse of Theorem 7.3.2 true? The converse is: If a sequence is bounded, then it converges. Clearly a counterexample to the last implication is the sequence $(-1)^n, n \in \mathbb{N}$. This sequence is bounded but it is not convergent.

The next question is whether boundedness and an additional property of a sequence can guarantee convergence. It turns out that such an property is monotonicity defined in the following definition.

Definition 7.3.3. A sequence $\{s_n\}_{n=1}^{+\infty}$ of real numbers is said to be

non-decreasing if $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$,

strictly increasing if $s_n < s_{n+1}$ for all $n \in \mathbb{N}$,

non-increasing if $s_n \geq s_{n+1}$ for all $n \in \mathbb{N}$.

strictly decreasing if $s_n > s_{n+1}$ for all $n \in \mathbb{N}$.

A sequence with either of these four properties is said to be *monotonic*.

The following two theorems give powerful tools for establishing convergence of a sequence.

Theorem 7.3.4. *If $\{s_n\}_{n=1}^{+\infty}$ is non-decreasing and bounded above, then $\{s_n\}_{n=1}^{+\infty}$ converges.*

Theorem 7.3.5. *If $\{s_n\}_{n=1}^{+\infty}$ is non-increasing and bounded below, then $\{s_n\}_{n=1}^{+\infty}$ converges.*

To prove these theorems we have to resort to the most important property of the set of real numbers: the Completeness Axiom.

The Completeness Axiom. If A and B are nonempty subsets of $\mathbb{R}$ such that for every $a \in A$ and for every $b \in B$ we have $a \leq b$, then there exists $c \in \mathbb{R}$ such that $a \leq c \leq b$ for all $a \in A$ and all $b \in B$.

Proof of Theorem 7.3.4. Assume that $\{s_n\}_{n=1}^{+\infty}$ is a non-decreasing sequence and that it is bounded above. Since $\{s_n\}_{n=1}^{+\infty}$ is non-decreasing we know that

$$s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_{n-1} \leq s_n \leq s_{n+1} \leq \cdots . \tag{7.3.1}$$

Let A be the range of the sequence $\{s_n\}_{n=1}^{+\infty}$. That is $A = \big\{s_n : n \in \mathbb{N}\big\}$. Clearly $A \neq \emptyset$. Let B be the set of all upper bounds of the sequence $\{s_n\}_{n=1}^{+\infty}$. Since the sequence $\{s_n\}_{n=1}^{+\infty}$ is bounded above, the set B is not empty. Let $b \in B$ be arbitrary. Then b is an upper bound for $\{s_n\}_{n=1}^{+\infty}$. Therefore

$$s_n \leq b \qquad \text{for all} \qquad n \in \mathbb{N}.$$

By the definition of A this means

$$a \leq b \qquad \text{for all} \qquad a \in A.$$

Since $b \in B$ was arbitrary we have

$$a \leq b \quad \text{for all} \quad a \in A \quad \text{and for all} \quad b \in B.$$

By the Completeness Axiom there exists $c \in \mathbb{R}$ such that

$$s_n \leq c \leq b \qquad \text{for all} \quad n \in \mathbb{N} \quad \text{and for all} \quad b \in B. \tag{7.3.2}$$

Thus c is an upper bound for $\{s_n\}_{n=1}^{+\infty}$ and also $c \leq b$ for all upper bounds b of the sequence $\{s_n\}_{n=1}^{+\infty}$. Therefore, for an arbitrary $\epsilon > 0$ the number $c - \epsilon$ (which is $< c$) is not an upper bound of the sequence $\{s_n\}_{n=1}^{+\infty}$. Consequently, there exists a natural number $N(\epsilon)$ such that

$$c - \epsilon < s_{N(\epsilon)}. \tag{7.3.3}$$

Let $n \in \mathbb{N}$ be any natural number which is $> N(\epsilon)$. Then the inequalities (7.3.1) imply that

$$s_{N(\epsilon)} \leq s_n. \tag{7.3.4}$$

By (7.3.2) the number c is an upper bound of $\{s_n\}_{n=1}^{+\infty}$. Hence we have

$$s_n \leq c \qquad \text{for all} \qquad n \in \mathbb{N}. \tag{7.3.5}$$

Putting together the inequalities (7.3.3), (7.3.4) and (7.3.5) we conclude that

$$c - \epsilon < s_n \leq c \quad \text{for all} \quad n \in \mathbb{N} \ \text{ such that } \ n > N(\epsilon). \tag{7.3.6}$$

The relationship (7.3.6) shows that for $n \in \mathbb{N}$ such that $n > N(\epsilon)$ the distance between numbers s_n and c is $< \epsilon$. In other words

$$n \in \mathbb{N}, \quad n > N(\epsilon) \quad \text{implies} \quad |s_n - c| < \epsilon.$$

This is exactly the implication in Definition 7.2.1. Thus, we proved that

$$\lim_{n \to +\infty} s_n = c.$$

□

Example 7.3.6. Prove that the sequence

$$t_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln n, \qquad n = 1, 2, 3, \ldots,$$

converges.

Solution. Use the definition of ln as the integral to prove that for $n > 1$

$$t_n > \int_1^n \left(\frac{1}{\mathrm{Floor}(x)} - \frac{1}{x} \right) dx.$$

Deduce that $t_n > 0$.

Represent

$$t_n - t_{n+1} = \bigl(\ln(n+1) - \ln n\bigr) - \frac{1}{n+1}$$

as an area (or a difference of two areas). Conclude that $t_n - t_{n+1} > 0$.

Now use one of the preceding theorems. □

8 Infinite series of real numbers

8.1 Definition and basic examples

The most important application of sequences is the definition of convergence of an infinite series. From the elementary school you have been dealing with addition of numbers. As you know the addition gets harder as you add more and more numbers. For example it would take some time to add

$$S_{100} = 1+2+3+4+5+\cdots+98+99+100$$

It gets much easier if you add two of these sums, and pair the numbers in a special way:

$$\begin{aligned} 2\,S_{100} = \quad & 1+\ \ 2+\ \ 3+\ \ 4+\cdots+97+98+99+100 \\ & 100+99+98+97+\cdots+\ \ 4+\ \ 3+\ \ 2+\ \ \ 1 \end{aligned}$$

A straightforward observation that each column on the right adds to 101 and that there are 100 such columns yields that

$$2\,S_{100} = 101\cdot 100, \qquad \text{that is} \qquad S_{100} = \frac{101\cdot 100}{2} = 5050.$$

This can be generalized to any natural number n to get the formula

$$S_n = 1+2+3+4+5+\cdots+(n-1)+n = \frac{(n+1)\,n}{2}.$$

This procedure indicates that it would be impossible to find the sum

$$1+2+3+4+5+\cdots+n+\cdots$$

where the last set of $\cdots$ indicates that we continue to add natural numbers.

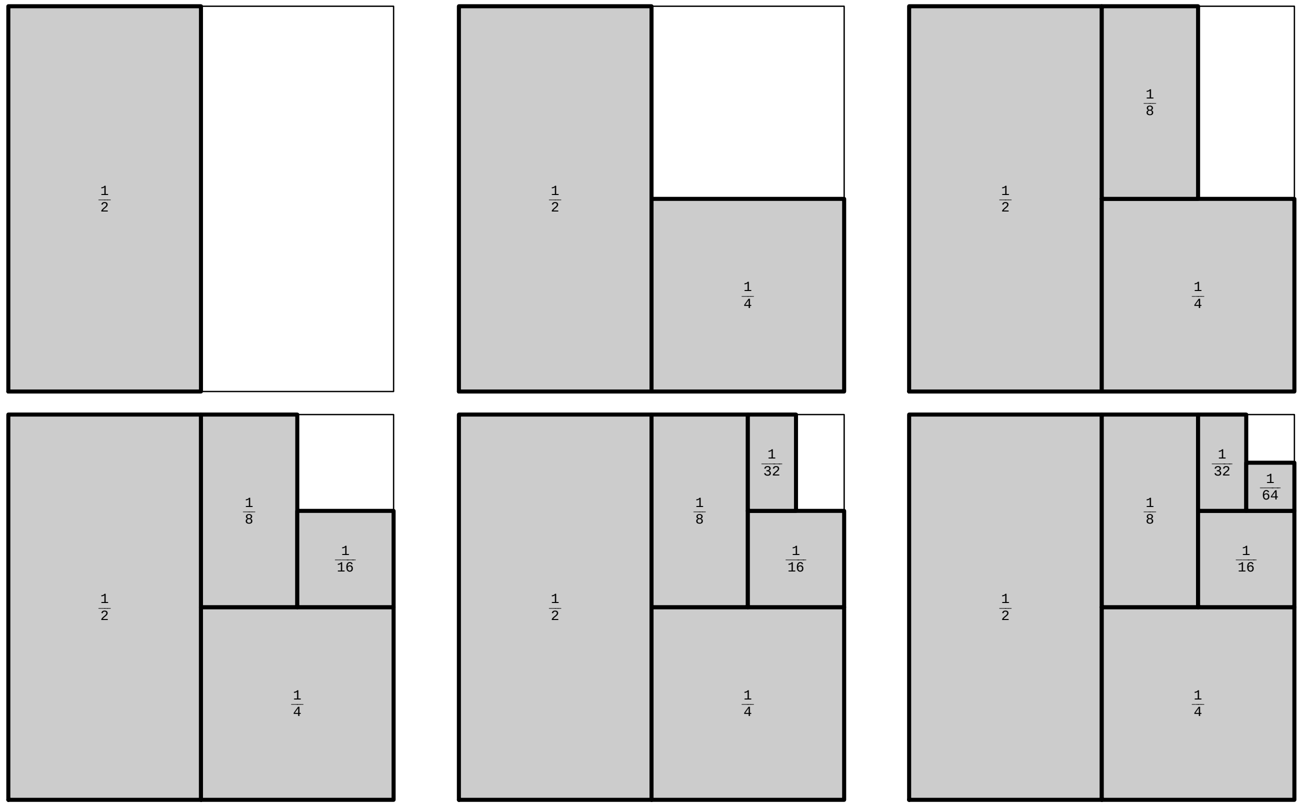
The situation is quite different if we consider the sequence

$$\frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \ldots,\ \frac{1}{2^n},\ \ldots$$

and start adding more and more consecutive terms of this sequence:

$$\begin{aligned}
&\frac{1}{2} && = 1-\frac{1}{2} = \frac{1}{2}\\
&\frac{1}{2}+\frac{1}{4} && = 1-\frac{1}{4} = \frac{3}{4}\\
&\frac{1}{2}+\frac{1}{4}+\frac{1}{8} && = 1-\frac{1}{8} = \frac{7}{8}\\
&\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} && = 1-\frac{1}{16} = \frac{15}{16}\\
&\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32} && = 1-\frac{1}{32} = \frac{31}{32}\\
&\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64} && = 1-\frac{1}{64} = \frac{63}{64}
\end{aligned}$$

These sums are nicely illustrated by the following pictures



In this example it seems natural to say that the sum of infinitely many numbers $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ equals 1:

$$\frac{1}{2} \;+\; \frac{1}{4} \;+\; \frac{1}{8} \;+\; \frac{1}{16} \;+\; \cdots \;+\; \frac{1}{2^n} \;+ \cdots \;=\; 1$$

Why does this make sense? This makes sense since we have seen above that as we add more and more terms of the sequence

$$\frac{1}{2},\; \frac{1}{4},\; \frac{1}{8},\; \frac{1}{16},\; \ldots,\; \frac{1}{2^n},\; \cdots$$

we are getting closer and closer to 1. Indeed,

$$\frac{1}{2} \;+\; \frac{1}{4} \;+\; \frac{1}{8} \;+\; \frac{1}{16} \;+\; \cdots \;+\; \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

and

$$\lim_{n \to +\infty} \left(1 - \frac{1}{2^n} \right) = 1.$$

This reasoning leads to the definition of convergence of an infinite series:

Definition 8.1.1. Let $\{a_n\}_{n=1}^{+\infty}$ be a given sequence. Then the expression

$$a_1 \;+\; a_2 \;+\; a_3 \;+\; \cdots \;+\; a_n \;+ \cdots$$

is called an infinite series. We often abbreviate it by writing

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{+\infty} a_n.$$

For each natural number n we calculate the (finite) sum of the first n terms of the series

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n.$$

We call S_n a partial sum of the infinite series $\displaystyle\sum_{n=1}^{+\infty} a_n$. (Notice that $\{S_n\}_{n=1}^{+\infty}$ is a new sequence.) If the sequence $\{S_n\}_{n=1}^{+\infty}$ converges and if

$$\lim_{n\to+\infty} S_n = S,$$

then the infinite series $\sum_{n=1}^{+\infty} a_n$ is called convergent and we write

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = S \qquad \text{or} \qquad \sum_{n=1}^{+\infty} a_n = S.$$

The number S is called the sum of the series.

If the sequence $\{S_n\}_{n=1}^{+\infty}$ does not converge, then the series is called divergent.

In the example above we have

$$\begin{aligned} a_n &= \frac{1}{2^n} = \left(\frac{1}{2}\right)^n, \\ S_n &= 1 - \frac{1}{2^n} = \frac{2^n - 1}{2^n} \\ \lim_{n\to+\infty} &\left(1 - \frac{1}{2^n}\right) = 1. \end{aligned}$$

Therefore we say that the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = \sum_{n=1}^{+\infty} \frac{1}{2^n}$$

converges and its sum is 1. We write $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{2^n} = 1$.

In our starting example

$$\begin{aligned} a_n &= n, \\ S_n &= 1 + 2 + 3 + \cdots + n = \frac{(n+1)n}{2} \\ \lim_{n\to+\infty} &\frac{(n+1)n}{2} \quad \text{does not exist.} \end{aligned}$$

Therefore we say that the series

$$1 \ + \ 2 \ + \ 3 \ + \ 4 \ + \ \cdots \ + \ n \ + \cdots \ = \sum_{n=1}^{+\infty} n$$

diverges.

Example 8.1.2 (Geometric Series)**.** Let a and r be real numbers. The most important infinite series is

$$a + a\,r + a\,r^2 + a\,r^3 + \cdots + a\,r^n + \cdots = \sum_{n=0}^{+\infty} a\,r^n \tag{8.1.1}$$

This series is called a geometric series. To determine whether this series converges or not we need to study its partial sums:

$$\begin{aligned}
&S_0 = a, && S_1 = a + a\,r,\\
&S_2 = a + a\,r + a\,r^2, && S_3 = a + a\,r + a\,r^2 + a\,r^3,\\
&S_4 = a + a\,r + a\,r^2 + a\,r^3 + a\,r^4, && S_5 = a + a\,r + a\,r^2 + a\,r^3 + a\,r^4 + a\,r^5,\\
&\quad\vdots && \\
&S_n = a + a\,r + a\,r^2 + \cdots + a\,r^{n-1} + a\,r^n && \\
&\quad\vdots &&
\end{aligned}$$

Notice that we have already studied the special case when $a = 1$ and $r = \dfrac{1}{2}$. In this special case we found a simple formula for S_n and then we evaluated $\lim\limits_{n\to+\infty} S_n$. It turns out that we can find a simple formula for S_n in the general case as well.

First note that the case $a = 0$ is not interesting, since then all the terms of the geometric series are equal to 0 and the series clearly converges and its sum is 0. Assume that $a \neq 0$. If $r = 1$ then $S_n = n\,a$. Since we assume that $a \neq 0$, $\lim\limits_{n\to+\infty} n\,a$ does not exist. Thus for $r = 1$ the series diverges.

Assume that $r \neq 1$. To find a simple formula for S_n, multiply the long formula for S_n above by r to get:

$$\begin{aligned}
S_n &= a + a\,r + a\,r^2 + \cdots + a\,r^{n-1} + a\,r^n,\\
r\,S_n &= a\,r + a\,r^2 + \cdots + a\,r^n + a\,r^{n+1};
\end{aligned}$$

now subtract,

$$S_n - r\,S_n = a - a\,r^{n+1},$$

and solve for S_n:

$$S_n = a\,\frac{1 - r^{n+1}}{1 - r}\,.$$

We already proved that if $|r|<1$, then $\lim_{n\to+\infty} r^{n+1}=0$. If $|r|\geq 1$, then $\lim_{n\to+\infty} r^{n+1}$ does not exist. Therefore we conclude that

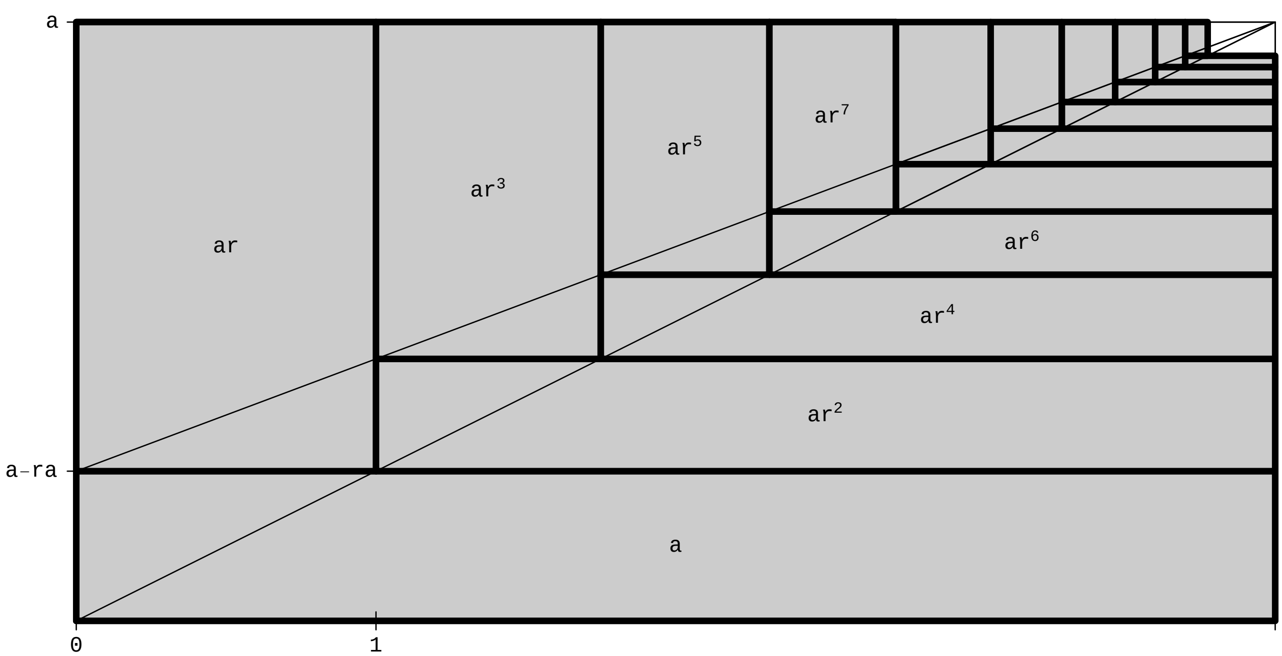
$$\begin{array}{lll} \lim_{n\to+\infty} S_n = \lim_{n\to+\infty} a\,\dfrac{1-r^{n+1}}{1-r} = a\,\dfrac{1}{1-r} & \text{for} & |r|<1, \\ \lim_{n\to+\infty} S_n \quad \text{does not exist} & \text{for} & |r|\geq 1\,. \end{array}$$

In conclusion

- If $|r|<1$, then the geometric series $\sum_{n=0}^{+\infty} a\,r^n$ converges and its sum is $a\,\dfrac{1}{1-r}$.

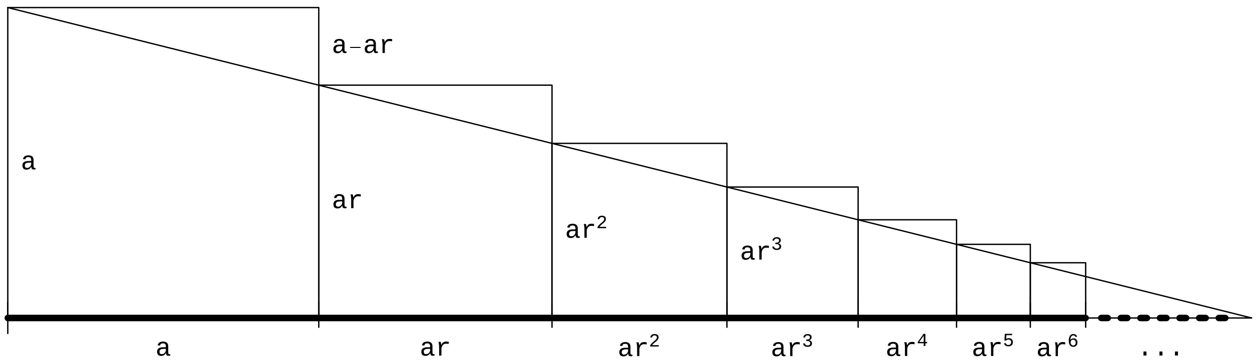
- If $|r|\geq 1$, then the geometric series $\sum_{n=0}^{+\infty} a\,r^n$ diverges.

The following picture illustrates the sum of a geometric series with $a>0$ and $0<r<1$. The width of the rectangle below is $1/(1-r)$ and the height is a. The slopes of the lines shown are $(1-r)a$ and $r(1-r)a$.



In the picture above the terms of a geometric series are represented as areas. As we can see the areas of the terms fill in the rectangle whose area is $a/(1-r)$.

In the picture below we represent the terms of the geometric series by lengths of horizontal line segments. The picture strongly indicates that the total length of infinitely many horizontal line segments is $a/(1-r)$. The reason for this is that by the construction the slope of the hypothenuse of the right triangle in the picture below is $-(1-r)$. Since its vertical leg is a, its horizontal leg must be $a(1-r)$.



Remark 8.1.3. **How to recognize whether an infinite series is a geometric series?**

Consider for example the infinite series $\displaystyle\sum_{n=1}^{+\infty} \frac{\pi^{n+2}}{e^{2n-1}}$. Here $a_n = \dfrac{\pi^{n+2}}{e^{2n-1}}$.

Looking at the formula (8.1.1) we note that the first term of the series is a and that the ratio between any two consecutive terms is r.

For $a_n = \dfrac{\pi^{n+2}}{e^{2n-1}}$ given above we calculate

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{\pi^{n+1+2}}{e^{2(n+1)-1}}}{\dfrac{\pi^{n+2}}{e^{2n-1}}} = \frac{\pi^{n+3}\, e^{2n-1}}{e^{2n+1}\, \pi^{n+2}} = \frac{\pi}{e^2}\,.$$

Since $\dfrac{a_{n+1}}{a_n}$ is constant, we conclude that the series $\displaystyle\sum_{n=1}^{+\infty} \frac{\pi^{n+2}}{e^{2n-1}}$ is a geometric series with

$$a = a_1 = \frac{\pi^2}{e} \quad \text{and} \quad r = \frac{\pi}{e^2} \quad \text{for all} \quad n = 1, 2, 3, \ldots .$$

Since $r = \dfrac{\pi}{e^2} < 1$, we conclude that the sum of this series is

$$\sum_{n=1}^{+\infty} \frac{\pi^{n+2}}{e^{2n-1}} = \frac{\pi^2}{e} \frac{1}{1 - \dfrac{\pi}{e^2}} = \frac{\pi^2}{e} \frac{e^2}{e^2 - \pi} = \frac{\pi^2\, e}{e^2 - \pi}\,.$$

Thus, to verify whether a given infinite series is a geometric series calculate the ratio of the consecutive terms and see whether it is a constant:

$$\sum_{n=1}^{+\infty} a_n \quad \text{for which} \quad \frac{a_{n+1}}{a_n} = r \quad \text{for all} \quad n = 1, 2, 3, \ldots \tag{8.1.2}$$

is a geometric series. In this case $a = a_1$ (the first term of the series).

Example 8.1.4 (Harmonic Series)**.** Harmonic series is the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots = \sum_{n=1}^{+\infty} \frac{1}{n}\,.$$

Again, to explore the convergence of this series we have to study its partial sums:

$$
\begin{aligned}
S_1 &= 1, & S_2 &= 1+\frac{1}{2},\\
S_3 &= 1+\frac{1}{2}+\frac{1}{3}, & S_4 &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4},\\
S_5 &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}, & S_6 &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6},\\
S_7 &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}, & S_8 &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8},\\
&\vdots\\
S_n &== 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}+\frac{1}{n}\\
&\vdots
\end{aligned}
$$

Since $S_{n+1}-S_n=\dfrac{1}{n+1}>0$ the sequence $\{S_n\}_{n=1}^{+\infty}$ is increasing.

Next we will prove that the sequence $\{S_n\}_{n=1}^{+\infty}$ is not bounded. We will consider only the natural numbers which are powers of 2: $\quad 2,4,8,\dots,2^k,\dots$. The following inequalities hold:

$$
\begin{aligned}
S_2 &= 1+\frac{1}{2} \geq 1+\frac{1}{2} && =1+1\,\frac{1}{2}\\
S_4 &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} \geq 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4} = 1+\frac{1}{2}+2\,\frac{1}{4} && =1+2\,\frac{1}{2}\\
S_8 &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\\
&\geq 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = 1+\frac{1}{2}+2\,\frac{1}{4}+4\,\frac{1}{8} && =1+3\,\frac{1}{2}\\
S_{16} &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}\\
&\geq 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = 1+\frac{1}{2}+2\,\frac{1}{4}+4\,\frac{1}{8}+8\,\frac{1}{16} && =1+4\,\frac{1}{2}
\end{aligned}
$$

Continuing this reasoning we conclude that for each $k=1,2,3,\dots$ the following formula holds:

$$
\begin{aligned}
S_{2^k} &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{8}+\cdots+\frac{1}{2^{k-1}}+\frac{1}{2^{k-1}+1}+\cdots+\frac{1}{2^k}\\
&\geq 1+\frac{1}{2}+2\,\frac{1}{4}+4\,\frac{1}{8}+8\,\frac{1}{16}+\cdots+2^{k-1}\,\frac{1}{2^k} && =1+k\,\frac{1}{2}
\end{aligned}
$$

Thus

$$S_{2^k} \geq 1+k\,\frac{1}{2} \quad \text{for all} \quad k=1,2,3,\dots \tag{8.1.3}$$

This formula implies that the sequence $\{S_n\}_{n=1}^{+\infty}$ is not bounded. Namely, let M be an arbitrary real number. We put $j=\max\bigl\{2\,\mathrm{Floor}(M),1\bigr\}$. Then

$$j \geq 2\,\mathrm{Floor}(M) > 2(M-1).$$

Therefore,

$$1 + j\,\frac{1}{2} > M.$$

Together with the inequality (8.1.3) this implies that

$$S_{2^j} > M.$$

Thus for an arbitrary real number M there exists a natural number $n = 2^j$ such that $S_n > M$. This proves that the sequence $\{S_n\}_{n=1}^{+\infty}$ is not bounded and therefore it is not convergent.

In conclusion:

- The harmonic series diverges.

The next example is an example of a series for which we can find a simple formula for the sequence of its partial sums and easily explore the convergence of that sequence. Examples of this kind are called telescoping series.

Example 8.1.5. Prove that the series $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$ converges and find its sum.

Solution. We need to examine the series of partial sums of this series:

$$S_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}, \quad n = 1, 2, 3, \ldots .$$

It turns out that it is easy to find the sum S_n if we use the partial fraction decomposition for each of the terms of the series:

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} \quad \text{for all} \quad k = 1, 2, 3, \ldots .$$

Now we calculate:

$$\begin{aligned}
S_n &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n\,(n+1)} \\
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= 1 - \frac{1}{n+1}.
\end{aligned}$$

Thus $S_n = 1 - \dfrac{1}{n+1}$ for all $n = 1, 2, 3, \ldots$. Using the algebra of limits we conclude that

$$\lim_{n \to +\infty} S_n = \lim_{n \to +\infty} \left(1 - \frac{1}{n+1}\right) = 1.$$

Therefore the series $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n\,(n+1)}$ converges and its sum is 1:

$$\sum_{n=1}^{+\infty} \frac{1}{n\,(n+1)} = 1.$$

$\square$

Exercise 8.1.6. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

(a) $\displaystyle\sum_{n=1}^{+\infty} 6\left(\frac{2}{3}\right)^{n-1}$ (b) $\displaystyle\sum_{n=1}^{+\infty} \frac{(-2)^{n+3}}{5^{n-1}}$ (c) $\displaystyle\sum_{n=0}^{+\infty} \frac{(\sqrt{2})^n}{2^{n+1}}$ (d) $\displaystyle\sum_{n=1}^{+\infty} \frac{e^{n+3}}{\pi^{n-1}}$

(e) $\displaystyle\sum_{n=1}^{+\infty} \frac{2^{2n-1}}{\pi^n}$ (f) $\displaystyle\sum_{n=1}^{+\infty} \frac{5}{2\,n}$ (g) $\displaystyle\sum_{n=0}^{+\infty} (\sin 1)^n$ (h) $\displaystyle\sum_{n=0}^{+\infty} \frac{2}{n^2+4n+3}$

(i) $\displaystyle\sum_{n=0}^{+\infty} (\cos 1)^n$ (j) $\displaystyle\sum_{n=2}^{+\infty} \frac{2}{n^2-1}$ (k) $\displaystyle\sum_{n=0}^{+\infty} (\tan 1)^n$ (l) $\displaystyle\sum_{n=1}^{+\infty} \ln\left(1+\frac{1}{n}\right)$

A digit is a number from the set $\{0,1,2,3,4,5,6,7,8,9\}$. A decimal number with digits $d_1, d_2, d_3, \ldots, d_n, \ldots$ is in fact an infinite series:

$$0.d_1d_2d_3\ldots d_n\ldots \;=\; \sum_{n=1}^{+\infty} \frac{d_n}{10^n}\,.$$

Therefore each decimal number with digits that repeat leads to a geometric series. We use the following abbreviation:

$$0.\overline{d_1d_2d_3\ldots d_k} = 0.d_1d_2d_3\ldots d_kd_1d_2d_3\ldots d_kd_1d_2d_3\ldots d_kd_1d_2d_3\ldots d_k\ldots$$

Exercise 8.1.7. Express the number as a ratio of integers.

(a) $0.\overline{9} = 0.999\ldots$ (b) $0.\overline{7} = 0.777\ldots$ (c) $0.\overline{712}$ (d) $0.\overline{5432}$

8.2 Basic properties of infinite series

An immediate consequence of the definition of a convergent series is the following theorem

Theorem 8.2.1. *If a series* $\displaystyle\sum_{n=1}^{+\infty} a_n$ *converges, then* $\displaystyle\lim_{n\to+\infty} a_n = 0$.

Proof. Assume that $\displaystyle\sum_{n=1}^{+\infty} a_n$ is a convergent series. By the definition of convergence of a series its sequence of partial sums $\{S_n\}_{n=1}^{+\infty}$ converges to some number S: $\displaystyle\lim_{n\to+\infty} S_n = S$. Then also $\displaystyle\lim_{n\to+\infty} S_{n-1} = S$. Now using the formula

$$a_n = S_n - S_{n-1}, \quad \text{for all} \quad n = 2,3,4,\ldots,$$

and the algebra of limits we conclude that

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} S_n - \lim_{n\to+\infty} S_{n-1} = S - S = 0.$$

□

Warning: The preceding theorem cannot be used to conclude that a particular series converges. Notice that in this theorem it is assumed that $\sum_{n=1}^{+\infty} a_n$ is a convergent.

On a positive note: Theorem 8.2.1 can be used to conclude that a given series diverges: If we know that $\lim_{n\to+\infty} a_n = 0$ is not true, then we can conclude that the series $\sum_{n=1}^{+\infty} a_n$ diverges. This is a useful test for divergence.

Theorem 8.2.2 (The Test for Divergence)**.** *If the sequence* $\{a_n\}_{n=1}^{+\infty}$ *does not converge to* 0*, then the series* $\sum_{n=1}^{+\infty} a_n$ *diverges.*

Example 8.2.3. Determine whether the infinite series $\sum_{n=1}^{+\infty} \cos\left(\frac{1}{n}\right)$ converges or diverges.

Solution. Just perform the divergence test:

$$\lim_{n\to+\infty} \cos\left(\frac{1}{n}\right) = 1 \neq 0\,.$$

Therefore the series $\sum_{n=1}^{+\infty} \cos\left(\frac{1}{n}\right)$ diverges. □

Example 8.2.4. Determine whether the infinite series $\sum_{n=1}^{+\infty} \frac{n^{(-1)^n}}{n+1}$ converges or diverges.

Solution. Consider the sequence $\left\{\frac{n^{(-1)^n}}{n+1}\right\}_{n=1}^{+\infty}$:

$$\frac{1}{1\cdot 2}, \frac{2}{3}, \frac{1}{3\cdot 4}, \frac{4}{5}, \frac{1}{5\cdot 6}, \frac{6}{7}, \frac{1}{7\cdot 8}, \frac{8}{9}, \frac{1}{9\cdot 10}, \frac{10}{11}, \frac{1}{11\cdot 12}, \frac{12}{13}, \ldots, \frac{1}{(2k-1)\cdot 2k}, \frac{2k}{2k+1}, \ldots \tag{8.2.1}$$

Without giving a formal proof we can tell that this sequence diverges. In my informal language the sequence (8.2.1) is not constantish since it can not decide whether to be close to 0 or 1.

Therefore the series $\sum_{n=1}^{+\infty} \frac{n^{(-1)^n}}{n+1}$ diverges. □

Remark 8.2.5. The divergence test can not be used to answer whether the series $\sum_{n=1}^{+\infty} \sin\left(\frac{1}{n}\right)$ converges or diverges. It is clear that $\lim_{n\to+\infty} \sin\left(\frac{1}{n}\right) = 0$. Thus we can not use the test for divergence.

Theorem 8.2.6 (The Algebra of Convergent Infinite Series)**.** *Assume that* $\sum_{n=1}^{+\infty} a_n$ *and* $\sum_{n=1}^{+\infty} b_n$ *are convergent series. Let* c *be a real number. Then the series*

$$\sum_{n=1}^{+\infty} c\, a_n, \qquad \sum_{n=1}^{+\infty} \bigl(a_n + b_n\bigr), \qquad \textit{and} \qquad \sum_{n=1}^{+\infty} \bigl(a_n - b_n\bigr),$$

are convergent series and the following formulas hold

$$\sum_{n=1}^{+\infty} c\, a_n = c \sum_{n=1}^{+\infty} a_n,$$
$$\sum_{n=1}^{+\infty} \bigl(a_n + b_n\bigr) = \sum_{n=1}^{+\infty} a_n + \sum_{n=1}^{+\infty} b_n, \quad \textit{and}$$
$$\sum_{n=1}^{+\infty} \bigl(a_n - b_n\bigr) = \sum_{n=1}^{+\infty} a_n - \sum_{n=1}^{+\infty} b_n\,.$$

Remark 8.2.7. The fact that we write $\sum_{n=1}^{+\infty} b_n$ does not necessarily mean that $\sum_{n=1}^{+\infty} b_n$ is a genuine infinite series.

For example, let m be a natural number and assume that $b_n = 0$ for all $n > m$. Then $\sum_{n=1}^{+\infty} b_n = \sum_{n=1}^{m} b_n$. In this case the series $\sum_{n=1}^{+\infty} b_n$ is clearly convergent. If $\sum_{n=1}^{+\infty} a_n$ is a convergent (genuine) infinite series, then Theorem 8.2.6 implies that the infinite series $\sum_{n=1}^{+\infty} \bigl(a_n + b_n\bigr)$ is convergent and

$$\sum_{n=1}^{+\infty} \bigl(a_n + b_n\bigr) = \sum_{n=1}^{+\infty} a_n + \sum_{n=1}^{m} b_n\,.$$

This in particular means that the nature of convergence of an infinite series can not be changed by changing finitely many terms of the series.

For example, let m be a natural number. Then:

$$\text{The series} \quad \sum_{n=1}^{+\infty} a_n \quad \text{converges if and only if the series} \quad \sum_{k=1}^{+\infty} a_{m+k} \quad \text{converges.}$$

Moreover, if $\sum_{n=1}^{+\infty} a_n$ converges, then the following formula holds

$$\sum_{n=1}^{+\infty} a_n = \sum_{j=1}^{m} a_j \; + \; \sum_{k=1}^{+\infty} a_{m+k}\,.$$

Example 8.2.8. Prove that the series $\displaystyle\sum_{n=1}^{+\infty}\left(\frac{\pi}{n(n+1)}-\frac{1}{2^n}\right)$ converges and find its sum.

Exercise 8.2.9. Determine whether the series is convergent or divergent. If a series is convergent find its sum.

(a) $\displaystyle\sum_{n=1}^{+\infty}\frac{n}{n+1}$ (b) $\displaystyle\sum_{n=1}^{+\infty}\arctan n$ (c) $\displaystyle\sum_{n=0}^{+\infty}\frac{3^n+2^n}{5^{n+1}}$ (d) $\displaystyle\sum_{n=2}^{+\infty}\left(\frac{3}{n^2-1}+\frac{\pi}{e^n}\right)$

(e) $\displaystyle\sum_{n=0}^{+\infty}\frac{e^n+\pi^n}{2^{2n-1}}$ (f) $\displaystyle\sum_{n=1}^{+\infty}n\sin\left(\frac{1}{n}\right)$ (g) $\displaystyle\sum_{n=0}^{+\infty}\frac{(n+1)^2}{n^2+1}$ (h) $\displaystyle\sum_{n=0}^{+\infty}\left((0.9)^n+(0.1)^n\right)$

Exercise 8.2.10. Express the following sums as ratios of integers and as repeating decimal numbers.

(a) $0.\overline{47}+0.\overline{5}$ (b) $0.\overline{499}+0.\overline{47}$ (c) $0.\overline{499}+0.\overline{503}$

8.3 Comparison Theorems

Warning: All series in the next two sections have positive terms! Do not use the tests from these sections for series with some negative terms.

The convergence of the series in Examples 8.1.2 and 8.1.5 was established by calculating the limits of their partial sums. This is not possible for most series. For example we will soon prove that the series

$$\sum_{n=1}^{+\infty}\frac{1}{n^2}$$

converges. To understand why the sum of this series is exactly $\dfrac{\pi^2}{6}$ you need to take a class about Fourier series, Math 430.

I hope that you have done your homework and that you proved that the series

$$\sum_{n=2}^{+\infty}\frac{1}{n^2-1}$$

converges and that you found its sum. If you didn't here is a way to do it: (It turns out that this is a telescoping series.)

Let

$$S_n=\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\cdots+\frac{1}{n^2-1}.$$

Since $S_{n+1}-S_n=\dfrac{1}{(n+1)^2-1}>0$ the sequence $\{S_n\}_{n=2}^{+\infty}$ is increasing.

For each $k=2,3,4,\ldots$ we have the following partial fractions decomposition

$$\frac{1}{k^2-1}=\frac{1}{(k-1)(k+1)}=\frac{1}{2}\left(\frac{1}{k-1}-\frac{1}{k+1}\right).$$

Next we use this formula to simplify the formula for the n-th partial sum

$$\begin{aligned} S_n &= \sum_{k=2}^{n} \frac{1}{k^2-1} = \sum_{k=2}^{n} \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right) = \frac{1}{2}\sum_{k=2}^{n}\left(\frac{1}{k-1} - \frac{1}{k+1}\right) \\ &= \frac{1}{2}\left(\left(\frac{1}{1}-\frac{1}{3}\right) + \left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \cdots + \left(\frac{1}{n-2}-\frac{1}{n}\right) + \left(\frac{1}{n-1}-\frac{1}{n+1}\right)\right) \\ &= \frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}\right) \\ &= \frac{1}{2}\left(\frac{3}{2}-\frac{2n+1}{n(n+1)}\right) = \frac{3}{4} - \frac{2n+1}{2n(n+1)}\,. \end{aligned}$$

Using the algebra of limits we calculate

$$\lim_{n\to+\infty} \frac{2n+1}{2n(n+1)} = \lim_{n\to+\infty} \frac{\dfrac{2n+1}{n^2}}{\dfrac{2n(n+1)}{n^2}} = \lim_{n\to+\infty} \frac{\dfrac{2}{n}+\dfrac{1}{n^2}}{2\dfrac{n+1}{n}} = \frac{0+0}{2\cdot 1} = 0\,.$$

Therefore, using the algebra of limits again, we calculate

$$\lim_{n\to+\infty} S_n = \frac{3}{4} - 0 = \frac{3}{4}\,.$$

Clearly $S_n < \dfrac{3}{4}$ for all $n = 2, 3, \ldots$.

Now consider the series

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$$

Let

$$T_n = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2}\,.$$

The fact that $T_{n+1} - T_n = \dfrac{1}{(n+1)^2} > 0$ implies that the sequence $\{T_n\}_{n=1}^{+\infty}$ is increasing.

Since

$$\frac{1}{4} < \frac{1}{3}, \quad \frac{1}{9} < \frac{1}{8}, \quad \frac{1}{16} < \frac{1}{15}, \quad \ldots, \quad \frac{1}{n^2} < \frac{1}{n^2-1},$$

we conclude that

$$T_n = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} < 1 + \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \cdots + \frac{1}{n^2-1} = 1 + S_n < 1 + \frac{3}{4}\,.$$

Thus $T_n < \dfrac{7}{4}$ for all $n = 2, 3, 4, \ldots$. Since the sequence $\{T_n\}_{n=1}^{+\infty}$ is increasing and bounded above it converges by Theorem 7.3.4. Thus the series $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2}$ converges and its sum is $< \dfrac{7}{4}$.

The principle demonstrated in the above example is the core of the following comparison theorem.

Theorem 8.3.1 (The Comparison Test)**.** *Let* $\sum_{n=1}^{+\infty} a_n$ *and* $\sum_{n=1}^{+\infty} b_n$ *be infinite series with positive terms. Assume that*

$$a_n \leq b_n \qquad \textit{for all} \quad n = 1, 2, 3, \ldots .$$

(a) *If* $\sum_{n=1}^{+\infty} b_n$ *converges, then* $\sum_{n=1}^{+\infty} a_n$ *converges and* $\sum_{n=1}^{+\infty} a_n \leq \sum_{n=1}^{+\infty} b_n$.

(b) *If* $\sum_{n=1}^{+\infty} a_n$ *diverges, then* $\sum_{n=1}^{+\infty} b_n$ *diverges.*

Sometimes the following comparison theorem is easier to use.

Theorem 8.3.2 (The Limit Comparison Test)**.** *Let* $\sum_{n=1}^{+\infty} a_n$ *and* $\sum_{n=1}^{+\infty} b_n$ *be infinite series with positive terms. Assume that*

$$\lim_{n \to +\infty} \frac{a_n}{b_n} = L.$$

If $\sum_{n=1}^{+\infty} b_n$ *converges, then* $\sum_{n=1}^{+\infty} a_n$ *converges. Or, equivalently, if* $\sum_{n=1}^{+\infty} a_n$ *diverges, then* $\sum_{n=1}^{+\infty} b_n$ *diverges.*

Example 8.3.3. Determine whether the series $\sum_{n=1}^{+\infty} \frac{n+1}{\sqrt{1+n^6}}$ converges or diverges.

Solution. The dominant term in the numerator is n and the dominant term in the denominator is $\sqrt{n^6} = n^3$. This suggests that this series behaves as the convergent series $\sum_{n=1}^{+\infty} \frac{1}{n^2}$. Since we are trying to prove convergence we will take

$$a_n = \frac{n+1}{\sqrt{1+n^6}} \qquad \text{and} \qquad b_n = \frac{1}{n^2}$$

in the Limit Comparison Test. Now calculate:

$$\lim_{n \to +\infty} \frac{\dfrac{n+1}{\sqrt{1+n^6}}}{\dfrac{1}{n^2}} = \lim_{n \to +\infty} \frac{n^2(n+1)}{\sqrt{1+n^6}} = \lim_{n \to +\infty} \frac{\dfrac{n^2(n+1)}{n^3}}{\dfrac{\sqrt{1+n^6}}{n^3}} = \lim_{n \to +\infty} \frac{1+\dfrac{1}{n}}{\sqrt{\dfrac{1}{n^6}+1}} = 1\,.$$

In the last step we used the algebra of limits and the fact that

$$\lim_{n \to +\infty} \sqrt{\frac{1}{n^6}+1} = 1$$

which needs a proof by definition.

Since we proved that $\lim_{n \to +\infty} \dfrac{\dfrac{n+1}{\sqrt{1+n^6}}}{\dfrac{1}{n^2}} = 1$ and since we know that $\sum_{n=1}^{+\infty} \dfrac{1}{n^2}$ is convergent, the Limit Comparison Test implies that the series $\sum_{n=1}^{+\infty} \dfrac{n+1}{\sqrt{1+n^6}}$ converges. □

In the next theorem we compare an infinite series with an improper integral of a positive function. Here it is presumed that we know how to determine the convergence or divergence of the improper integral involved.

Theorem 8.3.4 (The Integral Test). *Suppose that $x \mapsto f(x)$ is a continuous positive, decreasing function defined on the interval $(0, +\infty)$. Assume that $a_n = f(n)$ for all $n = 1, 2, \ldots$. Then the following statements are equivalent*

(a) *The integral $\int_1^{+\infty} f(x)\, dx$ converges.*

(b) *The series $\sum_{n=1}^{+\infty} a_n$ converges.*

At this point we assume that you are familiar with improper integrals and that you know how to decide whether an improper integral converges or diverges.

We will use this test in two different forms:

- Prove that the integral $\int_1^{+\infty} f(x)\, dx$ converges. Conclude that the series $\sum_{n=1}^{+\infty} a_n$ converges.
- Prove that the integral $\int_1^{+\infty} f(x)\, dx$ diverges. Conclude that the series $\sum_{n=1}^{+\infty} a_n$ diverges.

Example 8.3.5 (Convergence of p-series). Let p be a real number. The p-series $\sum_{n=1}^{+\infty} \dfrac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Solution. Let $n > 1$. Then the function $x \mapsto n^x$ is an increasing function. Therefore, if $p < 1$, then $n^p < n$. Consequently,

$$\frac{1}{n^p} > \frac{1}{n}, \qquad \text{for all} \quad n > 1 \quad \text{and} \quad p < 1\,.$$

Since the series $\sum_{n=1}^{+\infty} \dfrac{1}{n}$ diverges, the Comparison Test implies that the series $\sum_{n=1}^{+\infty} \dfrac{1}{n^p}$ diverges for all $p \leq 1$.

Now assume that $p > 1$. Consider the function $f(x) = \dfrac{1}{x^p}$, $x > 0$. This function is a continuous, decreasing, positive function. Let me calculate the improper integral involved in the Integral Test for convergence:

$$\int_1^{+\infty} \frac{1}{x^p}\,dx = \lim_{t\to+\infty} \int_1^t \frac{1}{x^p}\,dx = \lim_{t\to+\infty} \frac{1}{1-p}\,\frac{1}{x^{p-1}}\Bigg|_1^t$$

$$= \frac{1}{1-p}\lim_{t\to+\infty}\left(\frac{1}{t^{p-1}} - 1\right) = \frac{1}{1-p}(-1) = \frac{1}{p-1}$$

Thus this improper integral converges. Notice that the condition $p > 1$ was essential to conclude that $\displaystyle\lim_{t\to+\infty} \frac{1}{t^{p-1}} = 0$. Since $\dfrac{1}{n^p} = f(n)$ for all $n = 1, 2, 3, \ldots$, the Integral Test implies that the series $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^p}$ converges for $p > 1$. $\square$

Remark 8.3.6. We have not proved this for all $p > 1$ the function $f(x) = \dfrac{1}{x^p}$, $x > 0$, is continuous. One way to prove that for an arbitrary $a \in \mathbb{R}$ the function $x \mapsto x^a$, $x > 0$ is continuous is to use the identity

$$x^a = e^{a \ln x}, \qquad x > 0.$$

This identity shows that the function $x \mapsto x^a$, $x > 0$ is a composition of the function $\exp(x) = e^x$, $x \in \mathbb{R}$ and the function $x \mapsto a \ln x$, $x > 0$. The later function is continuous by the algebra of continuous functions: It is a product of a constant a and a continuous function ln. We proved that exp is continuous. By Theorem 6.2.3 a composition of continuous function is continuous. Consequently $x \mapsto x^a$, $x > 0$ is continuous.

Exercise 8.3.7. Determine whether the series is convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n\sqrt{n}}$ (b) $\displaystyle\sum_{n=1}^{+\infty} n e^{-n^2}$ (c) $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{n \ln n}$ (d) $\displaystyle\sum_{n=1}^{+\infty} \frac{\ln n}{n\sqrt{n}}$

(e) $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{n(\ln n)^b}$ (f) $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n!}$ (g) $\displaystyle\sum_{n=1}^{+\infty} \sin\bigl(\frac{1}{n}\bigr)$ (h) $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{n}\sin\bigl(\frac{1}{n}\bigr)$

(i) $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n}\cos\bigl(\frac{1}{n}\bigr)$ (j) $\displaystyle\sum_{n=0}^{+\infty} \frac{\pi + e^n}{e + \pi^n}$ (k) $\displaystyle\sum_{n=1}^{+\infty} \frac{n!}{n^n}$ (l) $\displaystyle\sum_{n=0}^{+\infty} \frac{n^2+1}{\sqrt{n^7+n^3+1}}$

For the series in (e) find all numbers b for which the series converges.

Exercise 8.3.8. A digit is a number from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. A decimal number with digits $d_1, d_2, d_3, \ldots, d_n, \ldots$ is in fact an infinite series:

$$0.d_1d_2d_3\ldots d_n\ldots = \sum_{n=1}^{+\infty} \frac{d_n}{10^n}.$$

Use a theorem from this section to prove that the series above always converges.

8.4 Ratio and root tests

Warning: All series in this section have positive terms! Do not use the tests from this section for series with negative terms.

In Remark 8.1.3 we pointed out (see (8.1.2)) that a series

$$\sum_{n=1}^{+\infty} a_n \quad \text{for which} \quad \frac{a_{n+1}}{a_n} = r \quad \text{for all} \quad n = 1, 2, 3, \ldots$$

is a **geometric series**. Consequently, if $|r| < 1$ this series is convergent, and it is divergent if $|r| \geq 1$.

Testing the series $\displaystyle\sum_{n=0}^{+\infty} \frac{1}{3^n - 2^{n+1}}$ using this criteria leads to the ratio

$$\frac{\dfrac{1}{3^{n+1} - 2^{n+2}}}{\dfrac{1}{3^n - 2^{n+1}}} = \frac{3^n - 2^{n+1}}{3^{n+1} - 2^{n+2}} = \frac{3^n\left(1 - 2\left(\dfrac{2}{3}\right)^n\right)}{3^{n+1}\left(1 - 2\left(\dfrac{2}{3}\right)^n\right)} = \frac{1}{3}\;\frac{1 - 2\left(\dfrac{2}{3}\right)^n}{1 - 2\left(\dfrac{2}{3}\right)^{n+1}}$$

which certainly is not constant, but it is "constantish." I propose that series for which the ratio a_{n+1}/a_n is not constant but constantish, should be called "geometrish." The following theorem tells that convergence and divergence of these series is determined similarly to geometric series.

Theorem 8.4.1 (The Ratio Test). *Assume that* $\displaystyle\sum_{n=1}^{+\infty} a_n$ *is a series with positive terms and that*

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = R\,.$$

Then

(a) *If* $R < 1$, *then the series converges.*

(b) *If* $R > 1$, *then the series diverges.*

Another way to recognize a geometric series is:

$$\text{A series} \quad \sum_{n=1}^{+\infty} a_n \quad \text{for which} \quad \sqrt[n]{\frac{a_{n+1}}{a_1}} = r \quad \text{for all} \quad n = 1, 2, 3, \ldots$$

is a **geometric series**. Consequently, if $|r| < 1$ this series is convergent, and it is divergent if $|r| \geq 1$.

Testing the series $\displaystyle\sum_{n=0}^{+\infty} \left(\frac{1+n}{1+2n}\right)^n$ using this criteria leads to the root

$$\sqrt[n]{\left(\frac{1+n}{1+2n}\right)^n} = \frac{1+n}{1+2n} = \frac{\frac{1}{n}+1}{\frac{1}{n}+2}$$

which certainly is not constant, but it is "constantish."

Theorem 8.4.2 (The Root Test)**.** *Assume that* $\sum_{n=1}^{+\infty} a_n$ *is a series with positive terms and that*

$$\lim_{n\to+\infty} \sqrt[n]{a_n} = R\,.$$

Then

(a) *If* $R < 1$, *then the series converges.*

(b) *If* $R > 1$, *then the series diverges.*

Remark 8.4.3. Notice that in both the ratio test and the root test if the limit $R = 1$ we can conclude neither divergence nor convergence. In this case the test is inconclusive.

Exercise 8.4.4. Determine whether the series is convergent or divergent.

(a) $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{2^n - 3}$	(b) $\displaystyle\sum_{n=1}^{+\infty} \left(\frac{n+2}{2n-1}\right)^n$	(c) $\displaystyle\sum_{n=1}^{+\infty} \frac{4^n}{3^{2n-1}}$	(d) $\displaystyle\sum_{n=1}^{+\infty} \frac{n!}{1\cdot 3\cdot 5\cdots(2n-1)}$
(e) $\displaystyle\sum_{n=1}^{+\infty} \frac{3^n n^2}{n!}$	(f) $\displaystyle\sum_{n=1}^{+\infty} e^{-n} n!$	(g) $\displaystyle\sum_{n=1}^{+\infty} \frac{e^{1/n}}{n^2}$	(h) $\displaystyle\sum_{n=1}^{+\infty} \frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n-1)}$
(i) $\displaystyle\sum_{n=1}^{+\infty} \frac{(n!)^2}{(2n)!}$	(j) $\displaystyle\sum_{n=1}^{+\infty} \frac{2\,n^{2n}}{(3n^2+1)^n}$	(k) $\displaystyle\sum_{n=1}^{+\infty} \frac{2^{3n}}{3^{2n}}$	(l) $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{(\arctan n)^n}$
(m) $\displaystyle\sum_{n=1}^{+\infty} \frac{n^2}{2^n}$	(n) $\displaystyle\sum_{n=1}^{+\infty} \frac{(n+1)^2}{n2^n}$	(o) $\displaystyle\sum_{n=1}^{+\infty} \frac{a^n}{n!}$	(p) $\displaystyle\sum_{n=1}^{+\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$

For some of the problems you might need to use tests from previous sections.