8.5 Alternating infinite series

In the previous two sections we considered only series with positive terms. In this section we consider series with both positive and negative terms which alternate: positive, negative, positive, etc. Such series are called **alternating series**. For example

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots+(-1)^{n+1}\frac{1}{n}+\cdots=\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{n} \tag{8.5.1}$$

$$1-1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}-\frac{1}{3}+\frac{1}{7}-\frac{1}{4}+\frac{1}{8}-\frac{1}{5}+\frac{1}{9}-\frac{1}{6}+\cdots=\sum_{n=1}^{+\infty}\frac{4(-1)^{n+1}}{n\big(3+(-1)^{n+1}\big)} \tag{8.5.2}$$

$$2-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\frac{6}{5}-\frac{7}{6}+\cdots+(-1)^{n+1}\frac{n+1}{n}+\cdots=\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{n+1}{n} \tag{8.5.3}$$

Theorem 8.5.1 (The Alternating Series Test)**.** *If the alternating series*

$$a_1-a_2+a_3-a_4+\cdots+(-1)^{n+1}a_n+\cdots=\sum_{n=1}^{+\infty}(-1)^{n+1}a_n$$

satisfies the following two conditions:

(i) $0<a_{n+1}\leq a_n$ *for all* $n=1,2,3,\ldots,$

(ii) $\displaystyle\lim_{n\to+\infty}a_n=0,$

then the series is convergent.

Proof. Assume that $\{a_n\}_{n=1}^{+\infty}$ is a non-increasing sequence (that is assume that (i) is true) and $\displaystyle\lim_{n\to+\infty}a_n=0$.

By the definition of convergence for each $\epsilon>0$ there exists $N(\epsilon)$ such that

$$n\in\mathbb{N},\quad n>N(\epsilon)\quad\text{implies}\quad |a_n-0|<\epsilon\,.$$

Since $a_n>0$, the last implication can be simplified as

$$n\in\mathbb{N},\quad n>N(\epsilon)\quad\text{implies}\quad a_n<\epsilon\,. \tag{8.5.4}$$

We need to show that the sequence of partial sums $\{S_n\}_{n=1}^{+\infty}$,

$$S_n=a_1-a_1-a_2+a_3-a_4+\cdots+(-1)^{n+1}a_n,\quad n=1,2,3,4,\ldots,$$

is convergent.

First consider the sequence $\{S_{2n}\}_{n=1}^{+\infty}$ of even partial sums. Then

$$S_{2(n+1)}-S_{2n}=a_{2n+1}-a_{2n+2}\geq 0,\quad\text{since by (i)}\quad a_{2n+2}\leq a_{2n+1}\,.$$

Thus the sequence $\{S_{2n}\}_{n=1}^{+\infty}$ is non-decreasing.

Next we compare an arbitrary even partial sum S_{2k} with an arbitrary odd partial sum S_{2j-1}. Assume $j \leq k$, then

$$S_{2k} - S_{2j-1} = \big(-a_{2j} + a_{2j+1}\big) + \big(-a_{2j+2} + a_{2j+3}\big) + \cdots + \big(-a_{2k-4} + a_{2k-3}\big) + \big(-a_{2k-2} + a_{2k-1}\big) - a_{2k}.$$

Each of the numbers in the parenthesis is negative. Therefore the last sum is negative. That is $S_{2k} \leq S_{2j-1}$ for $j \leq k$.

Assume now that $j > k$, then

$$S_{2j-1} - S_{2k} = \big(a_{2k+1} - a_{2k+2}\big) + \big(a_{2k+3} - a_{2k+4}\big) + \cdots + \big(a_{2j-5} - a_{2j-4}\big) + \big(a_{2j-3} - a_{2j-2}\big) + a_{2j-1}.$$

Each of the numbers in the parenthesis is positive. Therefore the last sum is positive. That is $S_{2k} \leq S_{2j-1}$ for $j > k$. Thus we conclude that

$$S_{2k} \leq S_{2j-1} \quad \text{for all} \quad j, k = 1, 2, 3, \dots . \tag{8.5.5}$$

In particular (8.5.5) means that $\{S_{2n}\}_{n=1}^{+\infty}$ is bounded above and that each S_{2j-1}, $j = 1, 2, 3, \dots$ is an upper bound. Since the sequence $\{S_{2n}\}_{n=1}^{+\infty}$ is also non-decreasing, the Monotone Convergence Theorem, Theorem 7.3.4, implies that $\{S_{2n}\}_{n=1}^{+\infty}$ converges to its least upper bound, call it S. Consequently

$$S_{2k} \leq S \leq S_{2j-1} \quad \text{for all} \quad j, k = 1, 2, 3, \dots . \tag{8.5.6}$$

For each two consecutive natural numbers $n, n-1$ one of them is even and one is odd. Therefore the inequalities in (8.5.6) imply that

$$|S_n - S| \leq |S_n - S_{n-1}| = a_n \quad \text{for all} \quad n = 1, 2, 3, \dots . \tag{8.5.7}$$

Let $\epsilon > 0$ be arbitrary. Let $n \in \mathbb{N}$ be such that $n > N(\epsilon)$. Then by (8.5.4) we conclude that

$$a_n < \epsilon \tag{8.5.8}$$

Combining the inequalities (8.5.7) and (8.5.8) we conclude that

$$|S_n - S| < \epsilon .$$

Thus we have proved that for each $\epsilon > 0$ there exists $N(\epsilon)$ such that

$$n \in \mathbb{N}, \quad n > N(\epsilon) \quad \text{implies} \quad |S_n - S| < \epsilon .$$

This proves that the sequence $\{S_n\}_{n=1}^{+\infty}$ converges and therefore the alternating series converges. $\square$

Example 8.5.2. The series in (8.5.1) is called alternating harmonic series. It converges.

Solution. We verify two conditions of the Alternating Series Test:

$$a_{n+1} \leq a_n \quad \text{since} \quad \frac{1}{n+1} < \frac{1}{n}, \quad \text{for all} \quad n = 1, 2, 3, \dots,$$

$$\lim_{n \to +\infty} \frac{1}{n} = 0 \quad \text{is easy to prove by definition.}$$

Thus the Alternating Series Test implies that the alternating harmonic series converges. $\square$

Remark 8.5.3. The Alternating Series Test does not apply to the series in (8.5.2) since the sequence of numbers

$$1,\ 1,\ \frac{1}{3},\ \frac{1}{2},\ \frac{1}{5},\ \frac{1}{3},\ \frac{1}{7},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{5},\ \frac{1}{9},\ \frac{1}{6},\ \ldots,\ \frac{4}{n\bigl(3+(-1)^{n+1}\bigr)},\ \ldots$$

is not non-increasing. Further exploration of the series in (8.5.2) would show that it diverges.

The Alternating Series Test does not apply to the series in (8.5.3) since this series does not satisfy the condition (ii):

$$\lim_{n\to+\infty}\frac{n+1}{n} = 1 \neq 0\,.$$

Again this series is divergent by the Test for Divergence.

Exercise 8.5.4. Determine whether the given series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{+\infty}\cos\Bigl(n\pi+\frac{1}{n}\Bigr)$ (b) $\displaystyle\sum_{n=0}^{+\infty}\sin\Bigl(n\frac{\pi}{2}\Bigr)$ (c) $\displaystyle\sum_{n=1}^{+\infty}\sin\Bigl(n\pi-\frac{1}{n}\Bigr)$

(d) $\displaystyle\sum_{n=1}^{+\infty}\frac{1}{n}\cos\Bigl(n\pi+\frac{1}{n}\Bigr)$ (e) $\displaystyle\sum_{n=1}^{+\infty}\ln\Bigl(1-\frac{(-1)^n}{n}\Bigr)$ (f) $\displaystyle\sum_{n=1}^{+\infty}\frac{1}{n}\sin\Bigl(n\frac{\pi}{2}\Bigr)$

(g) $\displaystyle\sum_{n=1}^{+\infty}\sin\Bigl(n\frac{\pi}{2}+\frac{1}{n}\Bigr)$ (h) $\displaystyle\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n-(-1)^n}$ (i) $\displaystyle\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{2n-(-1)^n}$

Many of the exercises in the next section use the Alternating Series Test for convergence. Do those exercises as well.

8.6 Absolute and Conditional Convergence

In the previous section we proved that the alternating harmonic series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots+(-1)^{n+1}\frac{1}{n}+\cdots=\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{n}\quad\text{converges.}\tag{8.6.1}$$

Later on we will see that the sum of this series is $\ln 2$.

Talking about infinite series in class I have often used the analogy with an infinite column in a spreadsheet and finding its sum. A series with positive and negative terms one can interpret as balancing a checkbook with (infinitely) many deposits and withdrawals. Looking at the alternating harmonic series (8.6.1) we see a sequence of alternating deposits and withdrawals, infinitely many of them. What we proved in the previous section tells that under two conditions on the deposits and withdrawals, although it has infinitely many transactions, this checkbook can be balanced.

Now comes the first surprising fact! Let's calculate how much has been deposited to this account:

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots+\frac{1}{2n-1}+\cdots=\sum_{n=1}^{+\infty}\frac{1}{2n-1}\,.$$

Applying the Limit Comparison Test with the harmonic series it is easy to conclude this series diverges. Looking at the withdrawals we see

$$-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\cdots-\frac{1}{2n-1}-\cdots=-\frac{1}{2}\sum_{n=1}^{+\infty}\frac{1}{n}.$$

Again this is a divergent series. This is certainly a suspicious situation: Dealing with an account to which an unbounded amount of money has been deposited and an unbounded amount of money has been withdrawn. A simpler way to look at this is to look at the total amount of money that went through this account (one can call this amount the total "activity" of the account):

$$\sum_{n=1}^{+\infty}\left|(-1)^{n+1}\frac{1}{n}\right| = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{n}+\cdots \tag{8.6.2}$$

This is the harmonic series which is divergent.

Since we know that an unbounded amount of money has been deposited to this account we might want to get in the spending mood sooner and do two withdrawals after each deposit, keeping the amounts the same:

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\frac{1}{7}-\frac{1}{14}-\frac{1}{16}+\cdots \tag{8.6.3}$$

In any real life checking account this might result in an occasional low balance but if the deposits and withdrawals are identical, no mater how you arrange them they should result in the same final balance. Amazingly this is not always the case with infinite series! (This is the second surprising fact!) The series in (8.6.3) also converges but to a different number then the series in (8.6.1). To be specific denote the terms of the series (8.6.3) by $b_n, n \in \mathbb{N}$. Then

$$b_{3k-2}=\frac{1}{2k-1}, \quad b_{3k-1}=-\frac{1}{4k-2}, \quad b_{3k}=-\frac{1}{4k}, \quad k\in\mathbb{N}.$$

It is clear that this series has the same terms as the alternating harmonic series. The terms of the alternating harmonic series has been reordered. For $k \in \mathbb{N}$, the term at the positions $2k-1$ (odd terms) in the alternating harmonic series is at the position $3k-2$ in the series (8.6.3), the term which is at the position $4k-2$ (a "half" of the even terms) in the alternating harmonic series is at the position $3k-1$ in the series (8.6.3) and the term which is at the position $4k$ (another "half" of the even terms) in the alternating harmonic series is at the position $3k$ in the series (8.6.3).

The following calculation indicates that the sum of the series in (8.6.3) is $1/2$ of the sum of the alternating harmonic series in (8.6.1). Let us calculate the $3n$-th partial sum of the series (8.6.3). Since this is a finite sum we can rearrange terms as we please. Here is the calculation

$$\begin{aligned}
S_{3n} &= 1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\cdots+\frac{1}{2n-1}-\frac{1}{4n-2}-\frac{1}{4n}\\
&= \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\cdots+\frac{1}{4n-2}-\frac{1}{4n}\\
&= \frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}\right)
\end{aligned}$$

Hence, $3n$-th partial sum of the series (8.6.3) is identical to one-half of the $2n$-th partial sum of the alternating harmonic series. Since the sum of the alternating harmonic series is $\ln 2$ we have

$$\lim_{n\to+\infty} S_{3n} = \frac{\ln 2}{2}.$$

Since

$$S_{3n+1} = S_{3n} + \frac{1}{2n+1} \quad \text{and} \quad S_{3n+2} = S_{3n} + \frac{1}{2n+1} - \frac{1}{4n+2} = S_{3n} + \frac{1}{4n+2},$$

we conclude that

$$\lim_{n\to+\infty} S_{3n+1} = \lim_{n\to+\infty} S_{3n+2} = \lim_{n\to+\infty} S_{3n} = \frac{\ln 2}{2}.$$

From the last three equalities one can prove rigorously that

$$\lim_{n\to+\infty} S_n = \frac{\ln 2}{2}.$$

This proves that the series (8.6.3) converges to $(\ln 2)/2$. That is just rearrangement of terms changed the sum.

This is a remarkable observation: a change of order of summation can change the sum of an infinite series. This feature is closely related to the fact that the total activity of the account expressed in (8.6.2) is a divergent series. This is a motivation for the following definition.

Definition 8.6.1. A convergent series $\sum_{n=1}^{+\infty} a_n$ is called **conditionally convergent** if the series of the absolute values of its terms $\sum_{n=1}^{+\infty} |a_n|$ is divergent.

Definition 8.6.2. A series $\sum_{n=1}^{+\infty} a_n$ is called **absolutely convergent** if the series of the absolute values of its terms $\sum_{n=1}^{+\infty} |a_n|$ is convergent.

Example 8.6.3. Prove that the series

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \cdots + (-1)^{n+1}\frac{1}{n^2} + \cdots = \sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{n^2}$$

is absolutely convergent.

Solution. By the definition of absolute convergence we need to determine the convergence of the series

$$\sum_{n=1}^{+\infty}\left|(-1)^{n+1}\frac{1}{n^2}\right| = \sum_{n=1}^{+\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots$$

This is a p-series with $p = 2$. Therefore this series converges. (Notice that at the beginning of Section 8.3 we proved that this series converges by comparing it to a telescoping series.) $\square$

Remark 8.6.4. One can interpreted the series in Example 8.6.3 as a checking account with infinitely many alternating deposits and withdrawals. In this case the total activity of the account is a convergent series. Consequently the total amount deposited

$$1+\frac{1}{9}+\frac{1}{25}+\cdots+\frac{1}{(2n-1)^2}+\cdots=\sum_{n=1}^{+\infty}\frac{1}{(2n-1)^2} \tag{8.6.4}$$

and the total amount withdrawn

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{36}+\cdots+\frac{1}{(2n)^2}+\cdots=\sum_{n=1}^{+\infty}\frac{1}{(2n)^2}=\frac{1}{4}\sum_{n=1}^{+\infty}\frac{1}{n^2} \tag{8.6.5}$$

are both convergent series. As we can see, the total amount withdrawn is 1/4 of the total activity of the account. We mentioned before that (we can not prove it in this course)

$$\sum_{n=1}^{+\infty}\frac{1}{n^2}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\cdots=\frac{\pi^2}{6}.$$

Therefore

$$\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{n^2}=1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\cdots=\frac{3}{4}\frac{\pi^2}{6}-\frac{1}{4}\frac{\pi^2}{6}=\frac{1}{2}\frac{\pi^2}{6}=\frac{\pi^2}{12}$$

Theorem 8.6.5. *If a series* $\sum_{n=1}^{+\infty} a_n$ *is absolutely convergent, then it is convergent.*

Proof. Assume that $\sum_{n=1}^{+\infty} a_n$ is absolutely convergent, that is assume that $\sum_{n=1}^{+\infty} |a_n|$ is convergent. Then the algebra of convergent series the series $\sum_{n=1}^{+\infty} 2\,|a_n|$ is convergent. Since $-|a_n|\leq a_n\leq |a_n|$, we conclude that

$$0\leq a_n+|a_n|\leq 2\,|a_n| \quad \text{for all} \quad n=1,2,3,\ldots$$

By the Comparison Test it follows that the series $\sum_{n=1}^{+\infty}\big(a_n+|a_n|\big)$ is convergent. The algebra of convergent series implies that the series

$$\sum_{n=1}^{+\infty}\Big(\big(a_n+|a_n|\big)-|a_n|\Big)=\sum_{n=1}^{+\infty}a_n$$

is also convergent. □

The following stronger versions of the Ratio and the Root test can be applied to any series to determine whether a series converges absolutely or it diverges.

Theorem 8.6.6 (The Ratio Test). *Let* $\sum_{n=1}^{+\infty} a_n$ *be a series for which* $\lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = R$. *Then*

(a) *If* $R < 1$, *then the series converges absolutely.*

(b) *If* $R > 1$, *then the series diverges.*

Theorem 8.6.7 (The Root Test). *Let* $\sum_{n=1}^{+\infty} a_n$ *be a series for which* $\lim_{n\to+\infty} \sqrt[n]{|a_n|} = R$. *Then*

(a) *If* $R < 1$, *then the series converges absolutely.*

(b) *If* $R > 1$, *then the series diverges.*

Notice that if the root or the ratio test apply to a series, then series either converges absolutely or diverges. This implies that if a series converges conditionally, then either

$$\lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = 1 \quad \text{or} \quad \lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} \text{ does not exist,}$$

and also

$$\lim_{n\to+\infty} \sqrt[n]{|a_n|} = 1 \quad \text{or} \quad \lim_{n\to+\infty} \sqrt[n]{|a_n|} \text{ does not exist.}$$

In other words, the root and the ratio test cannot lead to a conclusion that a series converges conditionally.

It turns out that our only tool which can be used to conclude conditional convergence is the alternating series test.

Exercise 8.6.8. Determine whether the given series converges conditionally, converges absolutely or diverges.

(a) $\sum_{n=0}^{+\infty} \frac{\cos(n\pi)}{n^2+1}$ (b) $\sum_{n=0}^{+\infty} \frac{\sin(n\pi/2)}{n+1}$ (c) $\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ (d) $\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$

(e) $\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^p}$ (f) $\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{e^{1/n}}{n}$ (g) $\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{n^n}{n!}$ (h) $\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{\sqrt{n}}{n+1}$

(i) $\sum_{n=2}^{+\infty} \frac{(-1)^{n+1}}{\ln n}$ (j) $\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{\ln n}{n}$ (k) $\sum_{n=1}^{+\infty} (-1)^{n+1} e^{1/n}$ (l) $\sum_{n=1}^{+\infty} (-1)^{n+1} \ln \frac{n+1}{n}$

In problem (e) determine all the values of p for which the series converges absolutely, converges conditionally and diverges.

Exercise 8.6.9. Determine whether the given series converges conditionally, converges absolutely or diverges.

(a) $\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{(\sin n)^2}{n^2}$ (b) $\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{4}{2n+3+(-1)^n}$

(c) $\sum_{n=1}^{+\infty} (-1)^{n+1} \cos\left(\frac{1}{n}\right)$ (d) $\sum_{n=1}^{+\infty} (-1)^{n+1} \sin\left(\frac{1}{n}\right)$

9 Series of functions

9.1 Power Series

The most important series is the **geometric series**:

$$a + a\,r + a\,r^2 + a\,r^3 + \cdots + a\,r^n + \cdots = \sum_{n=0}^{+\infty} a\,r^n\,.$$

If $-1 < r < 1$ the geometric series converges. Moreover, we proved

$$\sum_{n=0}^{+\infty} a\,r^n = a + a\,r + a\,r^2 + a\,r^3 + \cdots + a\,r^n + \cdots = \frac{a}{1-r} \quad \text{for} \quad -1 < r < 1\,. \tag{9.1.1}$$

Replacing r by x and letting $a = 1$ we can rewrite the formula in (9.1.1) as

$$\sum_{n=0}^{+\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \frac{1}{1-x} \quad \text{for} \quad -1 < x < 1\,. \tag{9.1.2}$$

The formula (9.1.2) can be viewed as a representation of the function

$$f(x) = \frac{1}{1-x}, \quad -1 < x < 1,$$

as an infinite series of powers of x: $1 = x^0, x, x^2, x^3, \ldots$:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{+\infty} x^n \quad \text{for} \quad -1 < x < 1\,.$$

You will agree that the (non-negative) integer powers of x are very simple functions. Therefore, it is natural to explore the following question:

Q1: Which functions can be represented as infinite series of constant multiples of (non-negative) integer powers of x?

In other words: Which functions $x \mapsto f(x)$ can be represented as

$$f(x) = a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + \cdots + a_n\,x^n + \cdots = \sum_{n=0}^{+\infty} a_n\,x^n \quad \text{for} \quad ? < x < ?\,.$$

The infinite series

$$a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + \cdots + a_n\,x^n + \cdots = \sum_{n=0}^{+\infty} a_n\,x^n \tag{9.1.3}$$

is called a **power series**.

The first question to answer about a power series is:

Q2: $\boxed{\text{For which real numbers } x \text{ does the power series converge?}}$

Since we are working with the powers of x and since there is no restriction on the signs of a_n and x, we can use Theorems 8.6.6 and 8.6.7 (the ratio and root test) to determine the absolute convergence of the power series (9.1.3). To apply Theorem 8.6.6 we calculate

$$\lim_{n\to+\infty} \frac{|a_{n+1}|\,|x|^{n+1}}{|a_n|\,|x|^n} = \lim_{n\to+\infty} \frac{|a_{n+1}|\,|x|}{|a_n|} = |x| \lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|}.$$

Assume that

$$\lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = L. \tag{9.1.4}$$

If $L = 0$, then Theorem 8.6.6 implies that the series (9.1.3) converges for all real numbers x. If $L > 0$, then Theorem 8.6.6 implies that the series (9.1.3)

$$\begin{aligned} \text{converges absolutely for} \quad & |x|\,L < 1, \quad \text{that is for} \quad -\frac{1}{L} < x < \frac{1}{L} \\ \text{diverges for} \quad & |x|\,L > 1, \quad \text{that is for} \quad x < -\frac{1}{L} \ \text{ or } \ x > \frac{1}{L} \end{aligned}$$

If the limit in (9.1.4) does not exist, then no conclusion about the convergence or divergence can be deduced.

To apply Theorem 8.6.7 we calculate

$$\lim_{n\to+\infty} \sqrt[n]{|a_n|\,|x|^n} = |x| \lim_{n\to+\infty} \sqrt[n]{|a_n|}.$$

Assume that

$$\lim_{n\to+\infty} \sqrt[n]{|a_n|} = L. \tag{9.1.5}$$

If $L = 0$, then Theorem 8.6.7 implies that the series (9.1.3) converges for all real numbers x. If $L > 0$, then Theorem 8.6.7 implies that the series (9.1.3)

$$\begin{aligned} \text{converges absolutely for} \quad & |x|\,L < 1, \quad \text{that is for} \quad -\frac{1}{L} < x < \frac{1}{L} \\ \text{diverges for} \quad & |x|\,L > 1, \quad \text{that is for} \quad x < -\frac{1}{L} \ \text{ or } \ x > \frac{1}{L} \end{aligned}$$

If the limit in (9.1.5) does not exist, then no conclusion about the convergence or divergence can be deduced.

Example 9.1.1. Consider the power series

$$\frac{1}{0!} + \frac{1}{1!}\,x + \frac{1}{2!}\,x^2 + \frac{1}{3!}\,x^3 + \cdots + \frac{1}{n!}\,x^n + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}\,x^n.$$

In this example $a_n = 1/n!$, $n = 0, 1, 2, \ldots$. We calculate

$$L = \lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to+\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to+\infty} \frac{1}{n+1} = 0.$$

Consequently the given power series converges absolutely for every $x \in \mathbb{R}$.

Example 9.1.2. Consider the power series

$$1 + 2x + 3\,x^2 + 4\,x^3 + \cdots + (n+1)\,x^n + \cdots = \sum_{n=0}^{\infty}(n+1)\,x^n.$$

Here $a_n = n+1$, $n = 0, 1, 2, \ldots$ and we calculate

$$L = \lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to+\infty} \frac{n+2}{n+1} = 1.$$

Consequently the given power series converges absolutely for every $x \in (-1, 1)$. Clearly the series diverges for $x = -1$ and for $x = 1$.

Example 9.1.3. Consider the power series

$$x - \frac{1}{2}\,x^2 + \frac{1}{3}\,x^3 - \frac{1}{4}\,x^4 + \cdots + (-1)^{n+1}\frac{1}{n}\,x^n + \cdots = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{n}\,x^n.$$

Here $a_0 = 0$ and $a_n = (-1)^{n+1}1/n$, $n = 1, 2, \ldots$. We calculate

$$L = \lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to+\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to+\infty} \frac{n}{n+1} = 1.$$

Consequently the given power series converges absolutely for every $x \in (-1, 1)$. Clearly the series diverges for $x = -1$ and converges conditionally for $x = 1$.

Example 9.1.4. Consider the power series

$$1 + \frac{1}{2}x + \frac{1}{2^2}\,x^2 + \frac{1}{2^3}\,x^3 + \cdots + \frac{1}{2^n}\,x^n + \cdots = \sum_{n=0}^{\infty} \frac{1}{2^n}\,x^n. \tag{9.1.6}$$

Here $a_n = 2^{-n}, n = 0, 1, 2, \ldots$. We calculate

$$L = \lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to+\infty} \frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}} = \lim_{n\to+\infty} \frac{1}{2} = \frac{1}{2}.$$

Consequently the given power series converges absolutely for every $x \in (-2, 2)$. Clearly the series diverges for $x = -2$ and for $x = 2$.

Notice that we can actually <u>calculate</u> the sum of this series using the following substitution (or you can call this a trick). Substitute $u = x/2$ in (9.1.6). Then (9.1.6) becomes

$$1 + u + u^2 + u^3 + \cdots + u^n + \cdots = \sum_{n=0}^{\infty} u^n. \tag{9.1.7}$$

We know that the sum of the series in (9.1.7) is $1/(1-u)$ for $u \in (-1, 1)$, that is,

$$1 + u + u^2 + u^3 + \cdots + u^n + \cdots = \sum_{n=0}^{\infty} u^n = \frac{1}{1-u}, \quad u \in (-1, 1).$$

Substituting back $u = x/2$ we get:

$$1 + \frac{1}{2}x + \frac{1}{2^2}\,x^2 + \frac{1}{2^3}\,x^3 + \cdots + \frac{1}{2^n}\,x^n + \cdots = \sum_{n=0}^{\infty} \frac{1}{2^n}\,x^n = \frac{2}{2-x}, \quad x \in (-2, 2).$$

Example 9.1.5. Consider the power series

$$\frac{1}{1}x + \frac{1}{4}\,x^2 + \frac{1}{9}\,x^3 + \cdots + \frac{1}{n^2}\,x^n + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}\,x^n.$$

We calculate

$$L = \lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to+\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to+\infty} \frac{n^2}{(n+1)^2} = 1.$$

Consequently the given power series converges absolutely for every $x \in (-1,1)$. For $x = 1$ we get the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Therefore, for $x = 1$ the given power series converges. For $x = -1$ we get the alternating series which converges absolutely. Therefore the given power series converges absolutely on $[-1,1]$.

The following theorem answers the question **Q2** above.

Theorem 9.1.6. *Let*

$$a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + \cdots + a_n\,x^n + \cdots = \sum_{n=0}^{+\infty} a_n\,x^n$$

be a power series. Then one of the following three cases holds.

(A) *The power series converges absolutely for all $x \in \mathbb{R}$.*

(B) *There exists $r > 0$ such that the power series converges absolutely for all $x \in (-r,r)$ and diverges for all x such that $|x| > r$.*

(C) *The power series diverges for all $x \neq 0$. For $x = 0$ it is trivial that the power series converges.*

The set on which a power series converges is called the *interval of convergence*. The number $r > 0$ in Theorem 9.1.6 (B) is called the *radius of convergence*. In the case (A) in Theorem 9.1.6 we write $r = +\infty$. In the case (C) in Theorem 9.1.6 we write $r = 0$.

Remark 9.1.7. In the case (B) in Theorem 9.1.6 the convergence of the power series at the points $x = r$ and $x = -r$ must be determined by studying the infinite series

$$\sum_{n=0}^{+\infty} a_n\,r^n \quad \text{and} \quad \sum_{n=0}^{+\infty} a_n\,(-r)^n.$$

A review of the examples in this section shows that the interval of convergence of a power series can have any of these four forms $(-r,r)$, $(-r,r]$, $[-r,r)$ and $[-r,r]$.

9.2 Functions Represented as Power Series

The following theorem lists properties of functions defined by a power series.

Theorem 9.2.1. *Let I be the interval of convergence of the power series*

$$a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + \cdots + a_n\,x^n + \cdots = \sum_{n=0}^{+\infty} a_n\,x^n.$$

Assume that I does not consist of a single point. Then the function f defined on I by

$$f(x) := a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + \cdots + a_n\,x^n + \cdots = \sum_{n=0}^{+\infty} a_n\,x^n, \quad x \in I,$$

has the following three properties.

(a) *The function f is continuous on I.*

(b) *The function f is differentiable at all interior points of I. Moreover,*

$$f'(x) = a_1 + 2a_2\,x + 3a_3\,x^2 + \cdots + na_n\,x^{n-1} + (n+1)a_{n+1}x^n + \cdots = \sum_{n=0}^{+\infty} (n+1)a_{n+1}\,x^n,$$

for all $x \in I$ excluding the endpoints (if any) of I.

(c) *The function f has derivatives of all orders $1, 2, 3, \ldots,$ at all interior points of I. In particular*

$$f(0) = a_0,\ f'(0) = a_1,\ f''(0) = 2\,a_2,\ f'''(0) = 3\cdot 2\,a_3,\ \ldots,\ f^{(n)}(0) = n!\,a_n,\ \ldots. \tag{9.2.1}$$

(d) *If $x \in I$, then*

$$\int_0^x f(t)dt = a_0x + \frac{a_1}{2}\,x^2 + \frac{a_2}{3}\,x^3 + \cdots + \frac{a_{n-1}}{n}\,x^n + \frac{a_n}{n+1}\,x^{n+1} + \cdots = \sum_{n=1}^{+\infty} \frac{a_{n-1}}{n}\,x^n.$$

Example 9.2.2. By (9.1.2) we have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \quad \text{for} \quad -1 < x < 1\,. \tag{9.2.2}$$

Thus the function $f(x) = 1/(1-x)$ defined for $x \in (-1, 1)$ can be represented by a power series. Applying Theorem 9.2.1 we get

$$\frac{1}{(1-x)^2} = 1 + 2\,x + 3\,x^2 + 4\,x^3 + \cdots + n\,x^{n-1} + (n+1)\,x^n + \cdots \quad \text{for} \quad -1 < x < 1\,.$$

Example 9.2.3. Substituting $-x$ for x in (9.2.2) we get

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots \quad \text{for} \quad -1 < x < 1\,. \tag{9.2.3}$$

Thus the function $f(x) = 1/(1+x)$ defined for $x \in (-1,1)$ can be represented by a power series. Applying Theorem 9.2.1 (d) we get

$$\ln(1+x) = \int_0^x \frac{1}{1+t}dt = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots + (-1)^{n+1}\frac{1}{n}x^n + \cdots \quad \text{for} \quad -1 < x < 1\,.$$

For $x = 1$ the above series is an alternating harmonic series which converges conditionally. Thus we found a power series representation for the function $\ln(1+x)$ on the interval $(-1,1]$. By Theorem 9.2.1 (a) this implies that the sum of the alternating harmonic series is $\ln 2$.

Example 9.2.4. Substituting x^2 for x in (9.2.3) we get

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots \quad \text{for} \quad -1 < x < 1\,.$$

Thus the function $f(x) = 1/(1+x^2)$ defined for $x \in (-1,1)$ can be represented by a power series. Applying Theorem 9.2.1 (d) we get

$$\arctan(x) = \int_0^x \frac{1}{1+t^2}dt = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots + (-1)^{n+1}\frac{1}{2n-1}x^{2n-1} + \cdots \quad \text{for} \quad -1 < x < 1\,.$$

For $x = 1$ the above series is a conditionally convergent alternating series. Moreover,

$$\frac{\pi}{4} = \arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^{n+1}\frac{1}{2n-1} + \cdots$$

Thus we have a power series representation for the function $\arctan(x)$ on the interval $(-1,1]$.

9.3 Taylor series at 0 (Maclaurin series)

In the preceding section we found power series representations for several well known functions. It turns out that all well known functions can be represented as power series. The key step in finding the power series representation of elementary functions are formulas (9.2.1) which establish the relationship between the coefficients a_n, $n = 0,1,2,\ldots$, of a power series and the derivatives of the function f which is represented by that power series. We rewrite formulas (9.2.1) as

$$a_0 = f(0), \quad a_1 = f'(0), \quad a_2 = \frac{1}{2!}f''(0), \quad a_3 = \frac{1}{3!}f^{(3)}(0), \ \ldots, \ a_n = \frac{1}{n!}f^{(n)}(0), \ldots. \tag{9.3.1}$$

Let $a > 0$ and let f be a function defined on $(-a,a)$. Assume that f has all derivatives on $(-a,a)$. Then the series power series

$$f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f^{(3)}(0)x^3 + \cdots + \frac{1}{n!}f^{(n)}(0)x^n + \cdots = \sum_{n=0}^{+infty} \frac{1}{n!}f^{(n)}(0)x^n$$

is called *Taylor series at* 0 or *Maclaurin series* of f.

Using formulas (9.3.1) it is not difficult to calculate a Maclaurin series for a given function. The difficulties arise in proving that the function defined by such power series is identical to the given function. Fortunately this is true for all well known functions.

Example 9.3.1. Let $f(x) = e^x = \exp(x)$, $x \in \mathbb{R}$. Then $f^{(n)}(x) = e^x$ for all $n = 0, 1, 2, \ldots$. Therefore the coefficients of the Maclaurin series for the function exp are $a_n = 1/n!$ and it can be proved that for all $x \in \mathbb{R}$ we have

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots .$$

Example 9.3.2. Let $f(x) = \sin(x)$, $x \in \mathbb{R}$. Then

$$f'(x) = \cos(x), \quad f''(x) = -\sin(x), \quad f^{(3)}(x) = -\cos(x), \quad f^{(4)}(x) = \sin(x).$$

Consequently,

$$f^{(2k)}(0) = 0, \quad f^{(2k+1)}(0) = (-1)^k, \quad k = 0, 1, 2, \ldots .$$

Therefore the coefficients of the Maclaurin series for the function sin are

$$a_{2k} = 0, \quad a_{2k+1} = (-1)^k \frac{1}{(2k+1)!}, \quad k = 0, 1, 2, \ldots .$$

It can be proved that for all $x \in \mathbb{R}$ we have

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots + (-1)^k \frac{1}{(2k+1)!}x^{2k+1} + \cdots .$$

Example 9.3.3. Let $f(x) = \cos(x)$, $x \in \mathbb{R}$. Then

$$f'(x) = -\sin(x), \quad f''(x) = -\cos(x), \quad f^{(3)}(x) = \sin(x), \quad f^{(4)}(x) = \cos(x).$$

Consequently,

$$f^{(2k)}(0) = (-1)^k, \quad f^{(2k+1)}(0) = 0, \quad k = 0, 1, 2, \ldots .$$

Therefore the coefficients of the Maclaurin series for the function cos are

$$a_{2k} = (-1)^k \frac{1}{(2k)!}, \quad a_{2k+1} = 0, \quad k = 0, 1, 2, \ldots .$$

It can be proved that for all $x \in \mathbb{R}$ we have

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + (-1)^k \frac{1}{(2k)!}x^{2k} + \cdots .$$

Example 9.3.4 (The Binomial Series)**.** Let $\alpha \in \mathbb{R}$. Let $f(x) = (1+x)^\alpha$, $x \in (-1, 1)$. Then

$$\begin{aligned}
f'(x) &= \alpha(1+x)^{\alpha-1}, \\
f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2}, \\
f^{(3)}(x) &= \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}, \\
&\ \vdots \\
f^{(n)}(x) &= \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n} \\
&\ \vdots
\end{aligned}$$

Therefore the coefficients of the Maclaurin series for the function f are

$$a_0 = 1, \quad a_n = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}, \quad n \in \mathbb{N}.$$

It can be proved that for all $x \in (-1,1)$ we have

$$(1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + \cdots.$$

This series is called *binomial series*. The reason for this name is that for $\alpha \in \mathbb{N}$ the binomial series becomes a polynomial:

$$\begin{aligned}
(1+x)^1 &= 1 + x \\
(1+x)^2 &= 1 + 2x + x^2 \\
(1+x)^3 &= 1 + 3x + 3x^2 + x^3 \\
(1+x)^4 &= 1 + 4x + 6x^2 + 4x^3 + x^4 \\
(1+x)^5 &= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 \\
(1+x)^6 &= 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6 \\
&\vdots \\
(1+x)^m &= \sum_{k=0}^{m} \binom{m}{k} x^k, \quad \text{were} \quad m \in \mathbb{N} \quad \text{and} \quad \binom{m}{k} = \frac{m!}{k!(m-k)!}
\end{aligned}$$

The last formula is called the *binomial theorem*. The coefficients

$$\binom{m}{k} = \frac{m!}{k!(m-k)!} = \frac{m(m-1)\cdots(m-k+1)}{k!} \quad \text{with} \quad m,k \in \mathbb{N}, \quad 0 \le k \le m,$$

are called *binomial coefficients*. With a general $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$ the coefficients

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

are called *generalized binomial coefficients*. By definition $\binom{\alpha}{0} = 1$. With this notation the binomial series can be written as

$$(1+x)^\alpha = \sum_{k=0}^{+\infty} \binom{\alpha}{k} x^k \quad \text{for} \quad x \in (-1,1). \tag{9.3.2}$$

Notice that formula (9.2.2) is a special case of (9.3.2), since

$$\binom{-1}{k} = \frac{(-1)(-2)\cdots(-1-k+1)}{k!} = \frac{(-1)^k k!}{k!} = (-1)^k.$$

Notice also that differentiating (9.2.2) leads to

$$(1+x)^{-2} = 1 + \sum_{k=1}^{+\infty} (-1)^k (k+1)\, x^k \quad \text{for} \quad -1 < x < 1\,.$$

This is a binomial series with $\alpha = -2$. To verify this we calculate

$$\binom{-2}{k} = \frac{(-2)(-3)\cdots(-2-k+1)}{k!} = \frac{(-1)^k(k+1)!}{k!} = (-1)^k(k+1).$$

For $\alpha = 1/2$ the expression

$$\begin{aligned}\binom{1/2}{k} &= \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(\frac{1}{2}-k+1\right)}{k!}\\ &= \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2k-3}{2}\right)}{k!}\\ &= \frac{(-1)^{k-1}1\cdot 3\cdot\ \cdots\ (2k-3)}{2^k\,k!}\end{aligned}$$

Thus

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{2^2 2!}x^2 + \frac{1\cdot 3}{2^3 3!}x^3 - \frac{1\cdot 3\cdot 5}{2^4 4!}x^4 + \frac{1\cdot 3\cdot 5\cdot 7}{2^5 5!}x^5 + \cdots \quad \text{for} \quad -1 < x < 1\,.$$

Example 9.3.5. Let $f(x) = \arcsin(x),\ x \in [-1,1]$. To calculate the Maclaurin series for arcsin we notice that

$$\frac{d}{dx}\big(\arcsin(x)\big) = \frac{1}{\sqrt{1-x^2}}, \qquad x \in (-1,1).$$

Now calculate the Maclaurin series for the last function using the binomial series with $\alpha = -1/2$. For $\alpha = -1/2$ and $k \in \mathbb{N}$, we calculate

$$\begin{aligned}\binom{-1/2}{k} &= \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-k+1\right)}{k!}\\ &= \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2k-1}{2}\right)}{k!}\\ &= (-1)^k\frac{1\cdot 3\cdot\ \cdots\ \cdot(2k-1)}{2^k\,k!}\end{aligned}$$

Thus

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{1\cdot 3}{2^2 2!}x^2 + \frac{1\cdot 3\cdot 5}{2^3 3!}x^3 - \frac{1\cdot 3\cdot 5\cdot 7}{2^4 4!}x^4 + \cdots \quad \text{for} \quad -1 < x < 1\,,$$

that is,

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{k=1}^{+\infty}(-1)^k\frac{1\cdot 3\cdot\ \cdots\ \cdot(2k-1)}{2^k\,k!}\ x^k,$$

or using the notation of double factorials

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{k=1}^{+\infty}(-1)^k\frac{(2k-1)!!}{(2k)!!}\ x^k.$$

Substituting $-x^2$ instead of x in the above formula we get

$$\frac{1}{\sqrt{1-x^2}} = 1 + \sum_{k=1}^{+\infty} \frac{(2k-1)!!}{(2k)!!}\ x^{2k}, \quad \text{for} \quad -1 < x < 1.$$

Since

$$\int_0^x \frac{1}{\sqrt{1-t^2}} dt = \arcsin(x),$$

integrating the last power series we get

$$\arcsin(x) = x + \sum_{k=1}^{+\infty} \frac{(2k-1)!!}{(2k+1)(2k)!!}\ x^{2k+1} = \sum_{k=0}^{+\infty} \frac{\binom{2k}{k}}{4^k(2k+1)}\ x^{2k+1}, \quad \text{for} \quad -1 < x < 1$$

It is interesting to note that the above expansion holds at both endpoints $x = -1$ and $x = 1$. To prove this we need to recall Theorem 9.2.1 (a) and prove that the series

$$1 + \sum_{k=1}^{+\infty} \frac{(2k-1)!!}{(2k+1)(2k)!!}$$

converges. This series converges by The Comparison Test. (**Hint:** Prove by mathematical induction that $\dfrac{(2k-1)!!}{(2k)!!} < \dfrac{1}{\sqrt[3]{k}}$ for all $k \in \mathbb{N}$.) As a consequence we obtain that

$$1 + \sum_{k=1}^{+\infty} \frac{(2k-1)!!}{(2k+1)(2k)!!} = \sum_{k=0}^{+\infty} \frac{\binom{2k}{k}}{4^k(2k+1)} = \frac{\pi}{2}.$$