Limits and Infinite Series

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March 14, 2014 11:03

Contents

CHAPTER 1

Limits

1. Numbers

All numbers in these notes are real numbers. The set of all real numbers is denoted by $\mathbb{R}$.

In this course we will use the standard set notation. We will be dealing with sets that consist of real numbers. A set can be described by a clear statement such as "Let A be the set of real solutions of the equation $x^2 - x = 0$." A set can also be described by a listing of all its elements; for example $A = \{0, 1\}$. To describe sets we often use the set builder notation:

$$A = \bigl\{x \in \mathbb{R} \,:\, x^2 = x\bigr\}.$$

The above expression is read as: "The set A is the set of all real numbers x such that $x^2 = x$." Here the colon ":" is used as an abbreviation for the phrase "such that". Instead of colon many books use the vertical bar symbol $|$.

Pay attention to the usage of the braces (or curly brackets) $\{$ and $\}$ in the set notation. The braces are used to delimit sets. Notice the difference: 0 is a real number. However, $\{0\}$ is the <u>set</u> whose only element is 0.

The expression $\bigl\{x \in \mathbb{R} \,:\, x^2 = x\bigr\}$ is read as "the set of all real numbers x such that $x^2 = x$".

The most important subsets of real numbers are the set of natural numbers, denoted by $\mathbb{N}$, and the set of integers, denoted by $\mathbb{Z}$. That is

$$\mathbb{N} = \{1, 2, 3, \ldots\}, \qquad \mathbb{Z} = \{-n : n \in \mathbb{N}\} \cup \{0\} \cup \mathbb{N}.$$

Here $\cup$ denotes the union of sets.

Important subsets of $\mathbb{R}$ are intervals. Let a and b be real numbers such that $a < b$. Here are all possible intervals with endpoints a and b.

$[a, b] = \bigl\{x \in \mathbb{R} \,:\, a \le x \le b\bigr\}$ is called *a closed interval*,
$(a, b) = \bigl\{x \in \mathbb{R} \,:\, a < x < b\bigr\}$ is called *an open interval*,
$[a, b) = \bigl\{x \in \mathbb{R} \,:\, a \le x < b\bigr\}$ is called *a half-open interval*,
$(a, b] = \bigl\{x \in \mathbb{R} \,:\, a < x \le b\bigr\}$ is called *a half-open interval*.

The intervals above are called *finite intervals*. We also define four types of *infinite intervals*:

$[a, +\infty) = \bigl\{x \in \mathbb{R} \,:\, a \le x\bigr\}$ is called *a closed unbounded interval*,
$(a, +\infty) = \bigl\{x \in \mathbb{R} \,:\, a < x\bigr\}$ is called *an open unbounded interval*
$(-\infty, b] = \bigl\{x \in \mathbb{R} \,:\, x \le b\bigr\}$ is called *an unbounded closed interval*,
$(-\infty, b) = \bigl\{x \in \mathbb{R} \,:\, x < b\bigr\}$ is called *an unbounded open interval.*

Geometric illustrations of these intervals are given in Figures 1 through 8.

FIG. 1. A closed interval

FIG. 2. An open interval

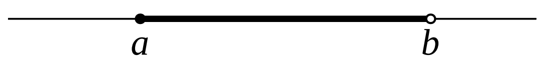

FIG. 3. A half-open interval

FIG. 4. A half-open interval

FIG. 5. A closed infinite interval

FIG. 6. An open infinite interval

FIG. 7. An infinite closed interval

FIG. 8. An infinite open interval

The infinity symbols $-\infty$ and $+\infty$ are used to indicate that the set is unbounded in the negative $(-\infty)$ or positive $(+\infty)$ direction of the real number line. The symbols $-\infty$ and $+\infty$ are just symbols; they are <u>not</u> real numbers. Therefore we always exclude them as endpoints by using parentheses.

The set $\mathbb{R}$ is also an infinite interval. Sometimes it is written as $(-\infty, +\infty)$.

Let S be a subset of $\mathbb{R}$. If u is the smallest number in S, then u is called a *minimum* of S and we write $u = \min S$. If v is the greatest number in S, then v is called a *maximum* of S and we write $v = \max S$. Notice that the set $\mathbb{Z}$ has neither a minimum nor a maximum. Also (a, b) has neither a minimum nor a maximum. The set $\mathbb{N}$ has no maximum and $\min \mathbb{N} = 1$. Each finite subset of $\mathbb{R}$ has both a minimum and a maximum.

2. Functions

2.1. The definition. Next we review the definition of a function. Let A and B be sets. A *function* f from A to B is a rule that assigns **exactly one** element of B to **each** element in A. This relationship between the sets A and B and the rule f is indicated by the following notation: $f : A \to B$. For $x \in A$ the unique element of B which is assigned to x by the function f is called the value of f at x. This element is denoted by $f(x)$. The set A is *domain* of f. The subset $\{f(x) \in B \ : \ x \in A\}$ of B is the *range* of f.

In this class we are interested in functions for which both sets A and B are subsets of the set of real numbers $\mathbb{R}$. Some examples of such functions are given next.

2.2. The sign and the unit step function. Let sign : $\mathbb{R} \to \mathbb{R}$ be given by the formula

$$\operatorname{sign}(x) := \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -1 & \text{for } x < 0. \end{cases}$$

This function is called the *sign* function.

Let us : $\mathbb{R} \to \mathbb{R}$ be given by the formula

$$\operatorname{us}(x) := \begin{cases} 1 & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

This function is called the *unit step* function.

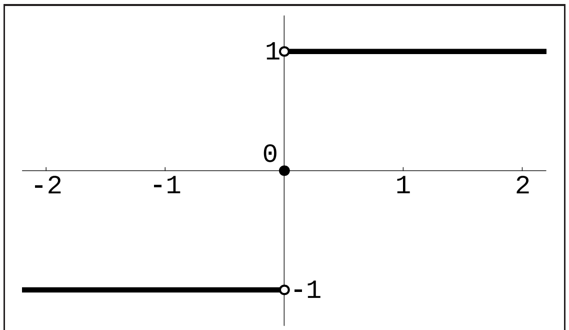

FIG. 9. The sign function

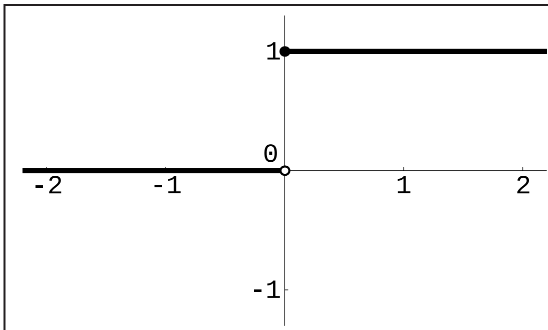

FIG. 10. The unit step function

EXERCISE 2.1. State clearly the domain and the range of the sign and the unit step function.

EXERCISE 2.2. Prove that $\max\{u, v\} = v + (u - v)\operatorname{us}(u - v)$ for all $u, v \in \mathbb{R}$.

2.3. The floor and the ceiling function. The *floor* function, floor : $\mathbb{R} \to \mathbb{R}$, is defined by the formula

$$\operatorname{floor}(x) = \lfloor x \rfloor = \max\bigl\{k \in \mathbb{Z} : k \leq x\bigr\}.$$

It follows from the properties of the maximum that for an arbitrary $x \in \mathbb{R}$ we have the following equivalence

$$m = \lfloor x \rfloor \qquad \text{if and only if} \qquad m \leq x < m + 1 \quad \text{and} \quad m \in \mathbb{Z}.$$

Notice that the inequalities $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$ are equivalent to

$$x - 1 < \lfloor x \rfloor \le x.$$

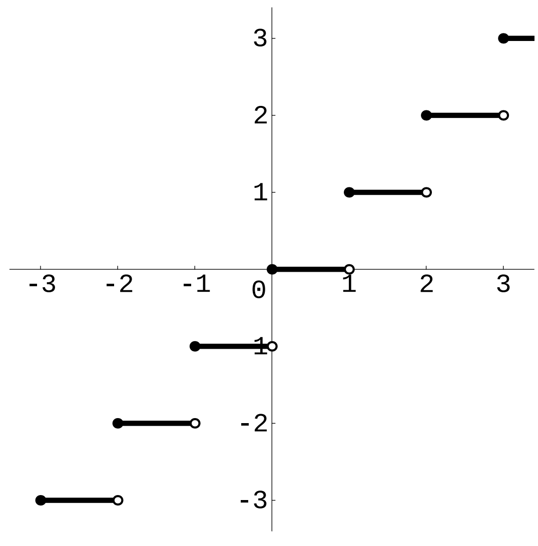

FIG. 11. The floor function

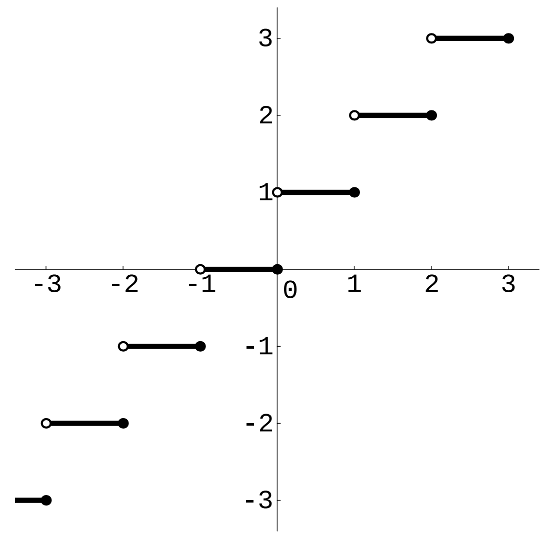

FIG. 12. The ceiling function

The *ceiling* function, ceiling : $\mathbb{R} \to \mathbb{R}$, is defined by the formula

$$\text{ceiling}(x) = \lceil x \rceil = \min\{k \in \mathbb{Z} : k \ge x\}.$$

It follows from the properties of the minimum that for an arbitrary $x \in \mathbb{R}$ we have the following equivalence

$$n = \lceil x \rceil \qquad \text{if and only if} \qquad n - 1 < x \le n \quad \text{and} \quad n \in \mathbb{Z}.$$

Notice that the inequalities $\lceil x \rceil - 1 < x \le \lceil x \rceil$ are equivalent to

$$x \le \lceil x \rceil < x + 1. \tag{2.1}$$

EXERCISE 2.3. State clearly the domain and the range of the floor and the ceiling function.

EXERCISE 2.4. Prove that for all $x \in \mathbb{R}$ we have

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \tfrac{1}{2} \rfloor.$$

Discover and prove the analogous identity for the ceiling function.

2.4. The absolute value function.

DEFINITION 2.5. Let abs : $\mathbb{R} \to \mathbb{R}$ be defined by the formula

$$\text{abs}(x) = |x| = \begin{cases} x & \text{if } x \ge 0, \\ -x & \text{if } x < 0. \end{cases}$$

This function is called the *absolute value* function. For a given real number x the number $|x|$ is called the *absolute value* of x.

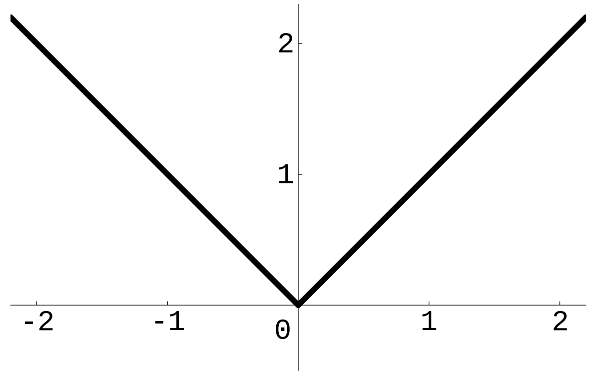

FIG. 13. The absolute value function

From calculus you are familiar with the geometric representation of real numbers as points on a straight line. This is done by choosing a point on the line to represent 0 and another point to represent 1. Then, every real number will correspond to a point on this line (called the number line), and every point on the number line will correspond to a real number. This geometric representation might be very helpful in doing problems.

Geometrically, the absolute value of a represents the distance between 0 and a, or, generally $|a - b|$ is the *distance* between the real numbers a and b on the number line.

EXERCISE 2.6. (a) Find all values of x such that $|5x - 3| = 4$.
(b) Find all values of x such that $|5x - 3| < 4$.
(c) Find all values of x such that $|5x - 3| > 4$.

EXERCISE 2.7. (a) Find all values of x such that $|7x + 3| = 5$.
(b) Find all values of x such that $|7x + 3| < 5$.
(c) Find all values of x such that $|7x + 3| > 5$.

The basic properties of the absolute value are given in the following exercises.

EXERCISE 2.8. Prove the following statements.

(i) For all $x \in \mathbb{R}$ we have $|x| = \max\{x, -x\}$.
(ii) $|x| \geq 0$ for all $x \in \mathbb{R}$.
(iii) $|-x| = |x|$ for all $x \in \mathbb{R}$.
(iv) $-x \leq |x|$ and $x \leq |x|$ for all $x \in \mathbb{R}$.
(v) $|xy| = |x||y|$ for all $x, y \in \mathbb{R}$.
(vi) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$ for all $x, y \in \mathbb{R}$, $y \neq 0$.

PROOF. To prove (i) we consider two cases. **Case I.** Assume $x \geq 0$. Then $-x \leq 0$. Since $-x \leq 0$ and $0 \leq x$, it follows that $-x \leq x$. Therefore $\max\{x, -x\} = x$. By Definition 2.5 for $x \geq 0$ we have that $\mathrm{abs}(x) = x$. Hence, we conclude that $\mathrm{abs}(x) = \max\{x, -x\}$ in this case. **Case II.** Assume $x < 0$. Then $-x > 0$. Since $-x > 0$ and $0 > x$, it follows that $-x > x$. Therefore $\max\{x, -x\} = -x$. By Definition 2.5 for $x < 0$ we have that $\mathrm{abs}(x) = -x$. Hence, we conclude that $\mathrm{abs}(x) = \max\{x, -x\}$ in this case.

Since Cases I and II include all real numbers x, the equality $\mathrm{abs}(x) = \max\{x, -x\}$ is proved.

The statement (ii) can also be proved by considering two cases.

To prove (iii) note that by (i) $|x| = \max\{x, -x\}$ and also $|-x| = \max\{-x, -(-x)\} = \max\{-x, x\}$. Since $\max\{x, -x\} = \max\{-x, x\}$, we conclude that $|x| = |-x|$.

By the definition of max we have $\max\{a, b\} \geq a$ and $\max\{a, b\} \geq b$ for any real numbers a and b. Therefore $\max\{x, -x\} \geq x$ and $\max\{x, -x\} \geq -x$. Using (i) we conclude $|x| \geq x$ and $|x| \geq -x$. This proves (iv). □

Exercise 2.9. Let x and y be real numbers. Prove that

$$\max\{x, y\} = \frac{1}{2}\big(x + y + |x - y|\big).$$

Exercise 2.10. Let $x \in \mathbb{R}$ and $a > 0$. Prove that $|x| < a$ if and only if $-a < x < a$.

Exercise 2.11. (a) Let $a, b \in \mathbb{R}$. Prove that $|a + b| \leq |a| + |b|$.
(b) Let $x, y, z \in \mathbb{R}$. Prove that $|x - y| \leq |x - z| + |z - y|$.
(c) Let $x, y \in \mathbb{R}$. Prove that $\big||x| - |y|\big| \leq |x - y|$.

Proof. Proof of (a). By Exercise 2.8 (iv) we know that $a \leq |a|$ and $b \leq |b|$. Therefore we conclude that

$$a + b \leq |a| + |b|. \tag{2.2}$$

By Exercise 2.8 (iv) we know that $-a \leq |a|$ and $-b \leq |b|$. Therefore we conclude

$$-(a + b) = -a + (-b) \leq |a| + |b|. \tag{2.3}$$

The inequalities (2.2) and (2.3) imply

$$\max\{a + b, -(a + b)\} \leq |a| + |b|. \tag{2.4}$$

By Exercise 2.8 (i) the inequality (2.4) yields $|a + b| \leq |a| + |b|$. This proves (a).

Prove (b) and (c) as an exercise. □

The inequalities in Exercise 2.11 are called the *triangle inequalities*.

Exercise 2.12. Let a, b, c be real numbers such that $a \neq 0$ and $c > 0$.

(a) Find all values of x such that $|ax + b| = c$.
(b) Find all values of x such that $|ax + b| < c$.
(c) Find all values of x such that $|ax + b| > c$.

Exercise 2.13. Let a be a real number and let ϵ be a positive real number. Prove that

$$|x - a| < \epsilon \qquad \text{if and only if} \qquad x \in (a - \epsilon, a + \epsilon).$$

2.5. New functions from old.

Definition 2.14. Given two functions $f : A \to B$ and $g : A \to B$, with $A, B \subset \mathbb{R}$, and two real numbers α and β we form a new function $\alpha f + \beta g : A \to B$ defined by

$$(\alpha f + \beta g)(x) := a\, f(x) + \beta\, g(x), \quad \text{for all} \quad x \in A.$$

Notice that $f(x)$ and $g(x)$ are real numbers so that $\alpha\, f(x)$ and $\beta\, g(x)$ in the above formula is just a multiplication of real numbers. The function $\alpha f + \beta g$ is called a *linear combination* of the functions f and g.

DEFINITION 2.15. Given two functions $f : A \to B$ and $g : A \to B$, with $A, B \subset \mathbb{R}$ we form a new function $fg : A \to B$ defined by

$$(fg)(x) := f(x)g(x), \quad \text{for all} \quad x \in A.$$

Notice that $f(x)$ and $g(x)$ are real numbers so that $f(x)g(x)$ in the above formula is just a multiplication of real numbers. The function fg is called the *product* of the functions f and g.

DEFINITION 2.16. Given two functions $f : A \to B$ and $g : B \to C$ a new function $g \circ f : A \to C$ is defined by

$$(g \circ f)(x) := g(f(x)), \qquad x \in A.$$

The function $g \circ f$ is called the *composition* of the functions f and g.

Applying these definitions to familiar functions gives rise to new, sometimes very interesting functions.

EXERCISE 2.17. For each of the functions given below answer the following questions: (a) What are the domain and the range of the function? (b) Plot the function using your graphing calculator. Plot the function by hand emphasizing the details missed by your graphing calculator.

(a) $x \mapsto x\,\mathrm{abs}(x)$	(b) $x \mapsto x(1 - \mathrm{abs}(x))$
(c) $x \mapsto x\,\mathrm{sign}(x)$	(d) $x \mapsto \mathrm{ceiling}(x) - \mathrm{floor}(x)$
(e) $x \mapsto x - \mathrm{floor}(x)$	(f) $x \mapsto x\,\mathrm{floor}(1/x)$
(g) $x \mapsto (1 + \mathrm{sign}(x))/2$	(h) $x \mapsto x\ \mathrm{us}(x)$
(i) $x \mapsto \mathrm{sign}(\mathrm{abs}(x))$	(j) $x \mapsto \mathrm{abs}(\mathrm{sign}(x))$
(k) $x \mapsto \mathrm{floor}(\mathrm{abs}(x))$	(l) $x \mapsto \mathrm{ceiling}(\mathrm{abs}(x))$

3. Limit of a function as x approaches $+\infty$

3.1. The definition.

DEFINITION 3.1. A function $x \mapsto f(x)$ has the limit L as x approaches $+\infty$ if the following two conditions are satisfied:

(I) There exists a real number X_0 such that $f(x)$ is defined for each $x \geq X_0$.
(II) For each real number $\epsilon > 0$ there exists a real number $X(\epsilon) \geq X_0$ such that

$$x > X(\epsilon) \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

If the conditions (I) and (II) in Definition 3.1 are satisfied we write $\lim_{x\to+\infty} f(x) = L$.

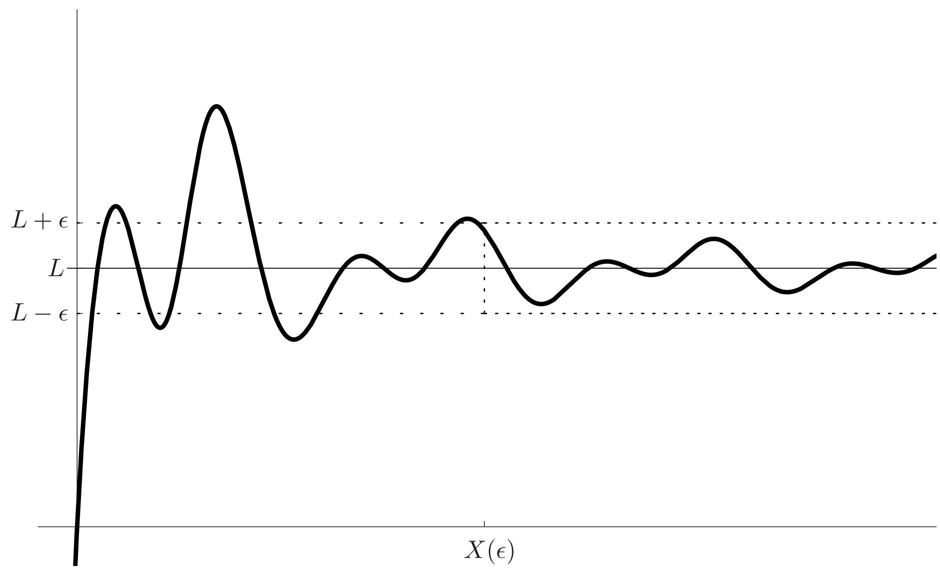

FIG. 14. An illustration for the condition (II) in Definition 3.1

3.2. Examples for Definition 3.1.

EXAMPLE 3.2. Prove that $\displaystyle\lim_{x\to+\infty} \frac{1}{\sqrt{x-1}} = 0$.

SOLUTION. We have to show that the conditions (I) and (II) in Definition 3.1 are satisfied. First we have to provide X_0. We can take $X_0 = 2$, since if $x \geq 2$, then $x - 1 > 0$ and $1/\sqrt{x-1}$ is defined.

Next we show that the condition (II) is satisfied. Let $\epsilon > 0$ be given. We have to find a real number $X(\epsilon) \geq 2$ such that

$$x > X(\epsilon) \quad \Rightarrow \quad \left|\frac{1}{\sqrt{x-1}} - 0\right| < \epsilon. \tag{3.1}$$

In some sense we have to solve the inequality

$$\left|\frac{1}{\sqrt{x-1}} - 0\right| < \epsilon.$$

for x. The first step is to simplify it. Clearly

$$\left|\frac{1}{\sqrt{x-1}} - 0\right| = \frac{1}{\sqrt{x-1}} \quad \text{for} \quad x \geq 2.$$

Thus we need to solve

$$\frac{1}{\sqrt{x-1}} < \epsilon.$$

This inequality is solved for x by using the following sequence of algebraic steps:

$$\frac{1}{\sqrt{x-1}} < \epsilon \quad \Leftrightarrow \quad \sqrt{x-1} > \frac{1}{\epsilon} \quad \Leftrightarrow \quad x-1 > \frac{1}{\epsilon^2} \quad \Leftrightarrow \quad x > \frac{1}{\epsilon^2} + 1. \tag{3.2}$$

Since we need $X(\epsilon) \geq 2$, we choose $X(\epsilon) := \max\left\{\frac{1}{\epsilon^2} + 1, 2\right\}$.

It remains to prove that the implication (3.1) is satisfied. Assume that

$$x > X(\epsilon). \tag{3.3}$$

Since $X(\epsilon) \geq 2$, we conclude that $x > 2$. Therefore $x - 1 > 0$ and $1/\sqrt{x-1}$ is defined. Since $X(\epsilon) \geq 1/\epsilon^2 + 1$, we conclude that

$$x > \frac{1}{\epsilon^2} + 1.$$

Now the equivalences (3.2) imply that

$$\frac{1}{\sqrt{x-1}} < \epsilon. \tag{3.4}$$

Since $1/\sqrt{x-1}$ is positive we conclude that

$$\frac{1}{\sqrt{x-1}} = \left|\frac{1}{\sqrt{x-1}}\right| = \left|\frac{1}{\sqrt{x-1}} - 0\right|. \tag{3.5}$$

Combining (3.4) and (3.5), yields

$$\left|\frac{1}{\sqrt{x-1}} - 0\right| < \epsilon. \tag{3.6}$$

Thus, we have proved that the assumption (3.3) implies the inequality (3.6). This is exactly the implication (3.1). □

Example 3.3. Determine the limit of the function $x \mapsto \dfrac{\text{ceiling}(x)}{x}$ as x approaches $+\infty$ and prove your claim.

Solution. In Subsection 2.3, see (2.1), we established that $x \leq \text{ceiling}(x) < x + 1$ for each real number x. Therefore, for large x, the value of ceiling(x) does not differ much from x. Therefore it is reasonable to make the following claim

$$\lim_{x \to +\infty} \frac{\text{ceiling}(x)}{x} = 1.$$

Next we will prove this claim using Definition 3.1. Since the function $x \mapsto \dfrac{\text{ceiling}(x)}{x}$ is defined for all $x \neq 0$, we can take $X_0 = 1$.

Next we show that the condition (II) is satisfied. Let $\epsilon > 0$ be given. We have to find a real number $X(\epsilon) \geq 1$ such that

$$x > X(\epsilon) \quad \Rightarrow \quad \left| \frac{\text{ceiling}(x)}{x} - 1 \right| < \epsilon. \tag{3.7}$$

Solving for x the inequality

$$\left| \frac{\text{ceiling}(x)}{x} - 1 \right| < \epsilon \tag{3.8}$$

is not easy. To find solutions of this inequality we first need to simplify it. In this process of simplification we can replace the expression

$$\left| \frac{\text{ceiling}(x)}{x} - 1 \right|$$

with an expression which is greater or equal to it. By the definition of the ceiling function we know that

$$x \leq \text{ceiling}(x) < x + 1. \tag{3.9}$$

Since we consider only $x \geq 1$, we can divide by x in (3.9) and subtract 1 from each term to get

$$0 \leq \frac{\text{ceiling}(x)}{x} - 1 < \frac{x+1}{x} - 1 = \frac{1}{x}.$$

Therefore

$$\left| \frac{\text{ceiling}(x)}{x} - 1 \right| \leq \frac{1}{x} \qquad \text{for all} \qquad x \geq 1. \tag{3.10}$$

This inequality is the key step in this proof. Therefore I call it the BIg INequality, or BIN. (Each of the proofs involving the definition of limit involves a BIN.) The importance of BIN lies in the fact that instead of solving (3.8), we can solve for x the simpler inequality

$$\frac{1}{x} < \epsilon.$$

The solution of this inequality (remember $x \geq 1$) is $x > \dfrac{1}{\epsilon}$.

Now we can define $X(\epsilon) := \max\left\{\dfrac{1}{\epsilon}, 1\right\}$. With this $X(\epsilon)$ the implication (3.7) is true. It is easy to prove this claim: Assume that

$$x > X(\epsilon) = \max\left\{\frac{1}{\epsilon}, 1\right\}.$$

Then $x \geq 1$ and $x > \dfrac{1}{\epsilon}$. Since $x \geq 1$ the BIN inequality (see (3.10))

$$\left| \frac{\text{ceiling}(x)}{x} - 1 \right| \leq \frac{1}{x}$$

is true. Since also $x > \dfrac{1}{\epsilon}$, we conclude that

$$\frac{1}{x} < \epsilon.$$

The last two displayed inequalities imply that

$$\left| \frac{\text{ceiling}(x)}{x} - 1 \right| < \epsilon.$$

This proves the implication (3.7). $\square$

EXERCISE 3.4. Determine whether the following functions have limits as x approaches $+\infty$. Prove your statements using the definition.

(a) $x \mapsto \dfrac{x}{3x-2}$ (b) $x \mapsto \dfrac{2x}{x^2+x+1}$ (c) $x \mapsto \dfrac{x+\sin(x)}{x-1}$

(d) $x \mapsto \dfrac{x^2+x}{x^3+3}$ (e) $x \mapsto \dfrac{x^3-2x^2+1}{x^3+x+101}$ (f) $x \mapsto \sqrt{x+1}-\sqrt{x}$

(g) $x \mapsto \dfrac{x^2+x\cos(x)}{x^2-x+5}$ (h) $x \mapsto \left(\dfrac{1}{x}\right)^{1/\ln x}$ (i) $x \mapsto \dfrac{x^2-1}{x^2+2x\sin(x)}$

(j) $x \mapsto x-\sqrt{x^2-x}$

EXERCISE 3.5. Guess the limit of the function $x \mapsto \ln\left(1+\dfrac{1}{x}\right)^x$ and prove your guess. Hint: 1) Use the rules for logarithms to simplify the expression. 2) Use the representation of the logarithm function $u \mapsto \ln(u)$ as an integral (area under the graph of the function $u \mapsto 1/u$) to find an upper and lower bound for the given function $x \mapsto \ln\left(1+\frac{1}{x}\right)^x$ for large values of x. The bounds should be very simple functions of x.

3.3. Negative results. How can we prove that $\lim_{x\to+\infty} f(x) = L$ is false? This means that the condition (I) or the condition (II) in Definition 3.1 is not satisfied.

Next we formulate the negation of the condition (I): (In class I will explain how to formulate negations of statements involving "for all" and "there exists")

The negation of (I): For each $X \in \mathbb{R}$ there exists $x \geq X$ such that $f(x)$ is not defined.

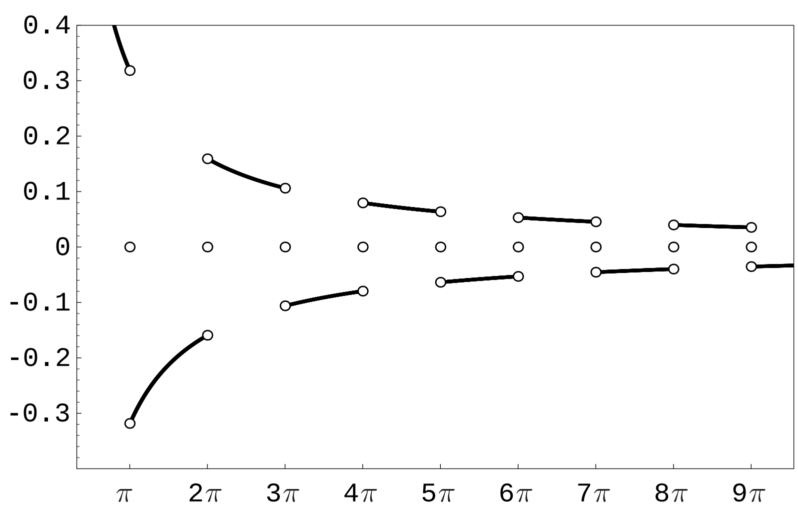

FIG. 15. This function does not satisfy (I) in Definition 3.1

EXAMPLE 3.6. Prove that the function $f(x) = \dfrac{1}{x\,\mathrm{sign}\bigl(\sin(x)\bigr)}$ does not satisfy the condition (I) in Definition 3.1.

SOLUTION. For this function the negation of (I) is true. This function is not defined for all $x = k\,\pi$ where $k \in \mathbb{Z}$. To prove that the negation of (I) is true let $X \in \mathbb{R}$ be arbitrary. Then

$$\pi\,\mathrm{ceiling}\bigl(X/\pi\bigr) \geq X$$

and $f(x)$ is not defined for $x = \pi\,\mathrm{ceiling}\bigl(X/\pi\bigr)$.

See Figure 15 for the graph of f. Small circles in the figure indicate that this function is not defined at $x = \pi, 2\pi, 3\pi, \ldots, 9\pi$. □

The negation of the condition (II) is more complicated.

The negation of (II): There exists $\epsilon > 0$ such that for every $X \in \mathbb{R}$ there exists $x > X$ such that $|f(x) - L| \geq \epsilon$.

EXAMPLE 3.7. Prove that $\displaystyle\lim_{x\to+\infty} \sin(x) = 0$ is false.

SOLUTION. Let $\epsilon = 1/2$. For arbitrary $X \in \mathbb{R}$ we have

$$\pi\,\mathrm{ceiling}\bigl(X/\pi\bigr) + \pi/2 > X$$

and, for $x = \pi\,\mathrm{ceiling}\bigl(X/\pi\bigr) + \pi/2$, we have $|\sin(x)| = 1$. Therefore

$$|\sin(x) - 0| \geq 1/2.$$

□

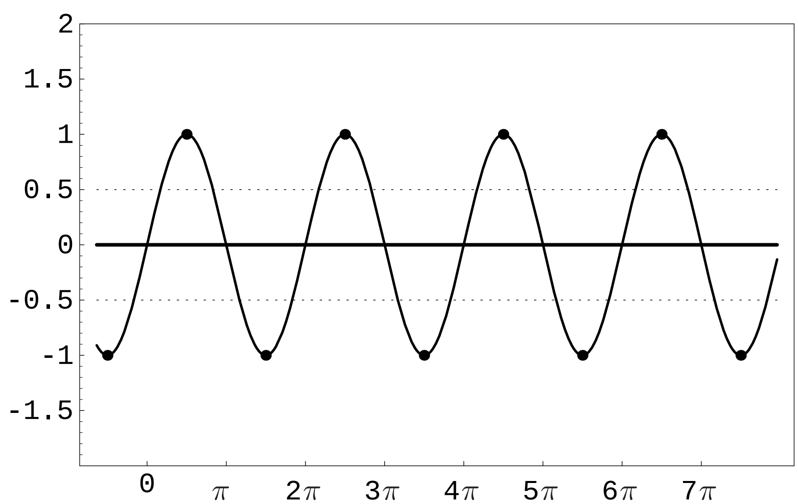

FIG. 16. Illustration for the solution of Example 3.7

Now we consider the statement "$\displaystyle\lim_{x\to+\infty} f(x)$ does not exist."

This means that for each $L \in \mathbb{R}$, $\displaystyle\lim_{x\to+\infty} f(x) = L$ is false.

EXAMPLE 3.8. Prove that $\displaystyle\lim_{x\to+\infty} \sin(x)$ does not exist.

SOLUTION. Let $L \in \mathbb{R}$ be arbitrary. We need to prove that $\lim\limits_{x\to+\infty} \sin(x) = L$ is false. Consider three cases $L = 0$, $L < 0$ and $L > 0$. The case $L = 0$ is done in Example 3.7. Now assume $L < 0$. Let $\epsilon = 1/2$. For arbitrary $X \in \mathbb{R}$ we have

$$2\pi \operatorname{ceiling}\left(\frac{X}{2\pi}\right) + \frac{\pi}{2} > X$$

and, for $x = 2\pi \operatorname{ceiling}\left(\frac{X}{2\pi}\right) + \frac{\pi}{2}$, we have $\sin(x) = 1$. Therefore

$$|\sin(x) - L| = |1 - L| = 1 + |L| \geq 1/2.$$

Do the case $L > 0$ as an exercise. □

3.4. Infinite limits.

DEFINITION 3.9. A function $x \mapsto f(x)$ has the limit $+\infty$ as x approaches $+\infty$ if the following two conditions are satisfied:

(I) There exists a real number X_0 such that $f(x)$ is defined for each $x \geq X_0$.
(II) For each real number M there exists a real number $X(M) \geq X_0$ such that

$$x > X(M) \quad \Rightarrow \quad f(x) > M.$$

In this case we write $\lim\limits_{x\to+\infty} f(x) = +\infty$.

DEFINITION 3.10. A function $x \mapsto f(x)$ has the limit $-\infty$ as x approaches $+\infty$ if the following two conditions are satisfied:

(I) There exists a real number X_0 such that $f(x)$ is defined for each $x \geq X_0$.
(II) For each real number M there exists a real number $X(M) \geq X_0$ such that

$$x > X(M) \quad \Rightarrow \quad f(x) < M.$$

3.5. Examples of infinite limits.

EXAMPLE 3.11. Let $f(x) = \sqrt{x}$. Prove that $\lim\limits_{x\to+\infty} \sqrt{x} = +\infty$.

SOLUTION. The function $\sqrt{\cdot}$ is defined for all $x \geq 0$. Therefore we can take $X_0 = 0$ in the part (I) of the definition.

Now consider the part (II) of the definition. Let $M \in \mathbb{R}$ be arbitrary. we have to determine a real number $X(M)$ such that

$$x > X(M) \quad \Rightarrow \quad \sqrt{x} > M.$$

This will be accomplished if we solve the inequality $\sqrt{x} > M$. If $M < 0$, then all $x \geq 0$ satisfy this inequality. If $M \geq 0$ then the solution of the inequality is $x > M^2$. Thus, we can take

$$X(M) = \begin{cases} M^2 & \text{if} \quad M \geq 0, \\ 0 & \text{if} \quad M < 0\,. \end{cases}$$

Clearly, $X(M) \geq 0$ for all $M \in \mathbb{R}$ and

$$x > X(M) \quad \Rightarrow \quad \sqrt{x} > M.$$

□

EXAMPLE 3.12. Let $f(x) = \operatorname{floor}(x)$. Prove that $\lim\limits_{x\to+\infty} \operatorname{floor}(x) = +\infty$.

SOLUTION. The function floor is defined for all $x \in \mathbb{R}$. Therefore we can take $X_0 = 0$ in the part (I) of the definition.

Now consider the part (II) of the definition. Let $M \in \mathbb{R}$ be arbitrary. We have to determine a real number $X(M) \geq X_0$ such that

$$x > X(M) \quad \Rightarrow \quad \text{floor}(x) > M. \tag{3.11}$$

This will be accomplished if we solve the inequality

$$\text{floor}(x) > M. \tag{3.12}$$

Since we don't know much about floor it is not easy to solve (3.12). To achieve the implication (3.11), we can replace floor(x) in (3.12) with a smaller quantity $g(x)$ such that $g(x) > M$ is easy to solve. Thus we need $g(x)$ such that

(A) floor$(x) \geq g(x)$ for all $x > X_0$.
(B) $g(x) > M$ is easy to solve.

By the definition of floor(x) we conclude that $0 \leq x - \text{floor}(x) < 1$ for all $x \in \mathbb{R}$. Therefore

$$x - 1 < \text{floor}(x) \quad \text{for all} \quad x \in \mathbb{R}. \tag{3.13}$$

Clearly $x - 1 > M$ is easy to solve: $x > M + 1$. Thus, we can take $X(M) = \max\{M + 1, 0\}$ in the part (II) of the definition. Clearly $X(M) \geq X_0 = 0$. Let $x > X(M)$. Then $x > M+1$ and therefore $x - 1 > M$. By the inequality (3.13) we conclude that

$$\text{floor}(x) > x - 1 > M.$$

Thus $x > X(M)$ implies floor$(x) > M$. □

The key step in the solution of Example 3.12 was the discovery of the function $g(x)$ such that

(A) $f(x) \geq g(x)$ for all $x > X_0$.
(B) $g(x) > M$ is easy to solve.

Most proofs about limits follow this same pattern. Therefore I refer to the discovery of the function g as a *Big Inequality* or BIN for short.

EXERCISE 3.13. Determine whether the following functions have the limit $+\infty$ when x approaches $+\infty$.

(a) $x \mapsto \dfrac{x^2}{2x+1}$ (b) $x \mapsto \ln x$ (c) $x \mapsto x - \sqrt{x}$

(d) $x \mapsto x - \ln(x)$ (e) $x \mapsto \dfrac{x^2 - x - 1}{x + 2\sqrt{x} + 1}$ (f) $x \mapsto \dfrac{1}{\sin\left(\frac{1}{x}\right)}$

(g) $x \mapsto \sqrt{x - \sqrt{x - \sqrt{x}}}$ (h) $x \mapsto \dfrac{(\cos x)^2 x}{\sqrt{x} + \sin(x)}$ (j) $x \mapsto \dfrac{(2 + \cos(x))x}{\sqrt{x} + \sin(x)}$

4. Limit of a function at a real number a

4.1. The definition.

DEFINITION 4.1. A function f has the limit $L \in \mathbb{R}$ as x approaches a real number a if the following two conditions are satisfied:

(I) There exists a real number $\delta_0 > 0$ such that $f(x)$ is defined for each x in the set $\bigl(a-\delta_0, a\bigr) \cup \bigl(a, a+\delta_0\bigr)$.

(II) For each real number $\epsilon > 0$ there exists a real number $\delta(\epsilon)$ such that $0 < \delta(\epsilon) \leq \delta_0$ and

$$0 < |x-a| < \delta(\epsilon) \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

REMARK 4.2. Notice that the condition that x belongs to the set $\bigl(a-\delta_0, a\bigr) \cup \bigl(a, a+\delta\bigr)$ can be expressed in terms of the distance between x and a as: $0 < |x-a| < \delta_0$.

Figure 17 illustrates Definition 4.1.

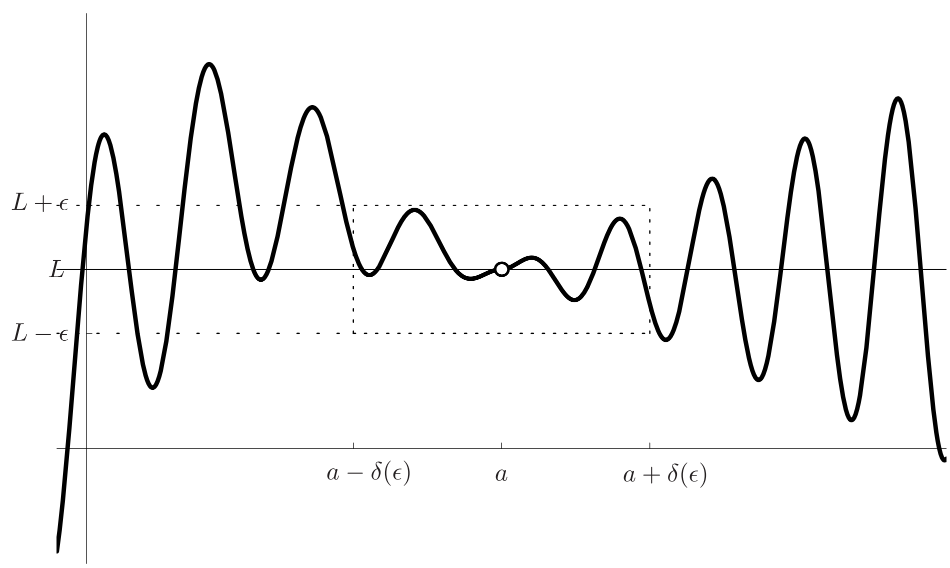

FIG. 17

Next we restate Definition 4.1 using the terminology of a calculator screen. The figure below shows a fictional calculator screen with 35 pixels. We assume that *ymin* and *ymax* are chosen in such a way that the number L is in the middle of the y-range and that *xmin* and *xmax* are such that a is in the middle of the x-range.

In Definition 4.3 below we assume that the function f satisfies (I) in Definition 4.1. We rephrase (II) from Definition 4.1 in terms of a calculator screen.

For the specific fictional calculator screen shown in Figure 18, the connection between Definition 4.1 and Definition 4.3 is given by $\epsilon = (ymax - ymin)/8$, $xmin = a - \delta(\epsilon)$, $xmax = a + \delta(\epsilon)$ and $\delta(\epsilon) = \Delta$.

The fictional screen in Figure 18 is chosen for its simplicity. The screen of TI-92 (see the manual p. 321) is 239 pixels wide and 103 pixels tall; it has 24617 pixels. The screen of TI-83 (see the manual p. 8-16) and of TI-82 is 95 pixels wide and 63 pixels tall; it has 5985 pixels. The screen of TI-85 (see the manual p. 4-13) is 127 pixels wide and 63 pixels tall; it has 8001 pixels. The screen of TI-89 (see the manual p. 222) is 159 pixels wide and

DEFINITION 4.3 (Calculator Screen). A function f has a limit L as x approaches a for each choice of $ymin$ and $ymax$ there exists Δ (which depends on $ymin$ and $ymax$) such that whenever we choose $xmin$ and $xmax$ such that $xmax - xmin < 2\Delta$ the graph of the function f will appear to be a straight horizontal line on the calculator screen with the only possible exception at the pixel containing $x = a$.

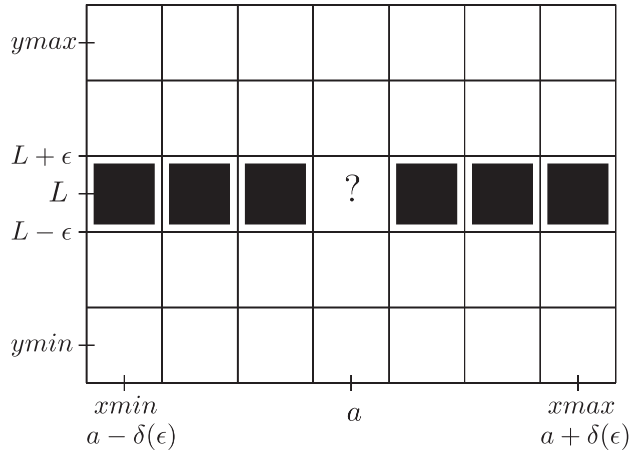

FIG. 18. A fictional calculator screen

77 pixels tall; it has 12243 pixels. Using these numbers you can calculate the connection between ϵ and $\delta(\epsilon)$ in Definition 4.1 and the screen of your calculator.

4.2. Examples for Definition 4.1.

EXAMPLE 4.4. Prove $\lim_{x\to 2}(3x-1) = 5$.

SOLUTION. (I) Here $f(x) = 3x - 1$. This function is defined on $\mathbb{R}$. We can take any positive number for δ_0. Since it might be useful to have a specific δ_0 to work with, we set $\delta_0 = 1$.

Let $\epsilon > 0$ be given. Let $\delta(\epsilon) = \min\{\epsilon/3, 1\}$. Assume $0 < |x-2| < \delta(\epsilon)$. Since $\delta(\epsilon) \leq \epsilon/3$, we conclude that $|x-2| < \epsilon/3$. Next, we calculate

$$|(3x-1)-5| = |3x-6| = 3\,|x-2|\,. \tag{4.1}$$

It follows from the assumption $0 < |x-2| < \delta(\epsilon)$ that $|x-2| < \epsilon/3$. Therefore we conclude

$$|(3x-1)-5| = 3\,|x-2| < 3\frac{\epsilon}{3} = \epsilon.$$

Thus we proved that

$$0 < |x-2| < \delta(\epsilon) \qquad \Rightarrow \qquad |(3x-1)-5| < \epsilon.$$

This is exactly the implication in (II) in Definition 4.1. Since $\epsilon > 0$ was arbitrary this completes the proof. □

REMARK 4.5. How did we guess the formula for $\delta(\epsilon)$ in the previous proof? We first studied the implication in the statement (II) in Definition 4.1. The goal in that implication is to prove

$$|(3x-1)-5| < \epsilon.$$

To prove this inequality we need to assume something about $|x-2|$. To find out what to assume, we simplified the expression $|(3x-1)-5|$ until $|x-2|$ appeared (see (4.1)). Then we solved for $|x-2|$. In this process of simplification we can afford to make the right-hand side larger. This will be illustrated in the next example.

EXAMPLE 4.6. Prove $\lim_{x\to 2}\left(3x^2 - 2x - 1\right) = 7$.

SOLUTION. As usual, we first deal with (I). Again $f(x) = 3x^2 - 2x - 1$ is defined on $\mathbb{R}$ and we can take any positive number for δ_0. Since it might be useful to have a specific choice of δ_0, we put $\delta_0 = 1$. (Notice that this implies that, from now on, we consider only in the values of x which are in the set $(1,2) \cup (2,3)$.)

Next we will discover an inequality which will help us find a formula for $\delta(\epsilon)$:

$$|(3x^2 - 2x - 1) - 7| = |3x^2 - 2x - 8| = |(3x+4)(x-2)| = |3x+4|\,|x-2|.$$

Now we use the fact that we are considering only the values of x which are in the set $(1,2) \cup (2,3)$. For $x \in (1,2) \cup (2,3)$ the value of $|3x+4|$ does not exceed 13. Therefore

$$|(3x^2 - 2x - 1) - 7| \leq 13\,|x-2| \quad \text{for all} \quad x \in (1,2) \cup (2,3).$$

Let $\epsilon > 0$ be given. The inequality $13\,|x-2| < \epsilon$ is easy to solve for $|x-2|$. The solution is $|x-2| < \epsilon/13$. Now we define $\delta(\epsilon)$:

$$\delta(\epsilon) = \min\left\{\frac{\epsilon}{13}, 1\right\}.$$

The remaining step of the proof is to prove the implication

$$|x-2| < \delta(\epsilon) \quad \Rightarrow \quad |(3x^2 - 2x - 1) - 7| < \epsilon.$$

We hope that at this point you can prove this implication on your own. □

EXAMPLE 4.7. Prove $\displaystyle\lim_{x\to 2} \frac{x^3 - x - 4}{x-1} = 2.$

SOLUTION. We first deal with (I). Notice that the function $f(x) = \dfrac{x^3 - x - 4}{x-1}$ is defined on $\mathbb{R}\setminus\{1\}$. In this proof we are interested in the values of x near $a = 2$. Therefore, for δ_0 we can take any positive number which is smaller than 1. Since it is useful to have a specific number, we put $\delta_0 = 1/2$. This implies that from now on we consider only the values of x which are in the set $(3/2, 2) \cup (2, 5/2)$.

Next we will discover an inequality which will help us find a formula for $\delta(\epsilon)$:

$$\left|\frac{x^3 - x - 4}{x-1} - 2\right| = \left|\frac{x^3 - 3x - 2}{x-1}\right| = \left|\frac{(x^2+2x+1)(x-2)}{x-1}\right| = \left|\frac{x^2+2x+1}{x-1}\right|\,|x-2|. \tag{4.2}$$

Now remember that we are interested only in the values of x which are in the set $(3/2, 2) \cup (2, 5/2)$. For $x \in (3/2, 2) \cup (2, 5/2)$ we estimate

$$\left|\frac{x^2+2x+1}{x-1}\right| = \frac{x^2+2x+1}{x-1} \leq \frac{16}{1/2} = 32 \quad \text{for all} \quad x \in (3/2, 2) \cup (2, 5/2). \tag{4.3}$$

Combining (4.2) and (4.3) we get

$$\left|\frac{x^3 - x - 4}{x-1} - 2\right| \leq 32\,|x-2| \quad \text{for all} \quad x \in (3/2, 2) \cup (2, 5/2).$$

Let $\epsilon > 0$ be given. The inequality $32\,|x-2| < \epsilon$ is very easy to solve for $|x-2|$. The solution is $|x-2| < \epsilon/32$. Now we define $\delta(\epsilon)$:

$$\delta(\epsilon) = \min\left\{\frac{\epsilon}{32}, \frac{1}{2}\right\}.$$

The remaining piece of the proof is to prove the implication

$$|x-2| < \delta(\epsilon) \quad \Rightarrow \quad \left|\frac{x^3 - x - 4}{x-1} - 2\right| < \epsilon.$$

We hope that at this point you can prove this on your own. Write down all the details of your reasoning. $\square$

EXAMPLE 4.8. Prove $\lim_{x\to 4}\sqrt{x}=2$.

SOLUTION. As usual, we first deal with (I). Notice that the function $f(x)=\sqrt{x}$ is defined on $(0,+\infty)$. We are interested in the values of x near the point $a=4$. Thus, for δ_0 we can take any positive number which is <4. Since it is useful to have a specific number, we put $\delta_0=1$. (Notice that this implies that from now on in this proof we are interested only in the values of x which are in the set $(3,4)\cup(4,5)$.)

Next we will discover an inequality which will help us find a formula for $\delta(\epsilon)$:

$$\left|\sqrt{x}-2\right|=\left|\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}+2}\right|=\left|\frac{x-4}{\sqrt{x}+2}\right|=\left|\frac{1}{\sqrt{x}+2}\right|\,|x-4|. \tag{4.4}$$

Now remember that we are interested only in the values of x which are in the set $(3,4)\cup(4,5)$. For $x\in(3,4)\cup(4,5)$ we estimate

$$\left|\frac{1}{\sqrt{x}+2}\right|=\frac{1}{\sqrt{x}+2}\leq\frac{1}{\sqrt{3}+2}\leq\frac{1}{2}\quad\text{for all}\quad x\in(3,4)\cup(4,5). \tag{4.5}$$

Combining (4.4) and (4.5) we get

$$\left|\sqrt{x}-2\right|\leq\frac{1}{2}\,|x-4|\quad\text{for all}\quad x\in(3,4)\cup(4,5).$$

Let $\epsilon>0$ be given. The inequality $\frac{1}{2}\,|x-4|<\epsilon$ is easy to solve for $|x-4|$. The solution is $|x-4|<2\epsilon$. Now define $\delta(\epsilon)$:

$$\delta(\epsilon)=\min\{2\epsilon,1\}.$$

The remaining step of the proof is to prove the implication

$$|x-4|<\min\{2\epsilon,1\}\quad\Rightarrow\quad\left|\sqrt{x}-2\right|<\epsilon.$$

We hope that at this point you can prove this on your own. As before, please do it and write down the details of your reasoning. $\square$

EXAMPLE 4.9. Prove that for any $a>0$, $\lim_{x\to a}\frac{1}{x}=\frac{1}{a}$.

SOLUTION. Let $a>0$. As before, we first deal with (I) in Definition 4.1. Notice that the function $f(x)=1/x$ is defined on $\mathbb{R}\setminus\{0\}$. We are interested in the values of x near the point $a>0$. Thus, for δ_0 we can take any positive number which is $<a$. Since it is useful to have a specific number, we put $\delta_0=a/2$. (Notice that this implies that from now on in this proof we are interested only in the values of x which are in the set $(a/2,a)\cup(a,3a/2)$.)

Next we will discover an inequality which will help us find a formula for $\delta(\epsilon)$:

$$\left|\frac{1}{x}-\frac{1}{a}\right|=\left|\frac{a-x}{xa}\right|=\frac{|a-x|}{xa}=\frac{1}{xa}\,|x-a|. \tag{4.6}$$

Now remember that we are interested only in the values of x which are in the set $(a/2,a)\cup(a,3a/2)$. For $x\in(a/2,a)\cup(a,3a/2)$ we estimate

$$\frac{1}{xa}\leq\frac{1}{(a/2)a}=\frac{2}{a^2}\quad\text{for all}\quad x\in(a/2,a)\cup(a,3a/2). \tag{4.7}$$

Combining (4.6) and (4.7) we get

$$\left|\frac{1}{x}-\frac{1}{a}\right| \leq \frac{2}{a^2}\,|x-a| \quad \text{for all} \quad x \in (a/2,a)\cup(a,3a/2).$$

Let $\epsilon > 0$ be given. The inequality $\frac{2}{a^2}\,|x-a| < \epsilon$ is easy to solve for $|x-a|$. The solution is $|x-a| < (a^2/2)\epsilon$. Now define $\delta(\epsilon)$:

$$\delta(\epsilon) = \min\left\{\frac{a^2\epsilon}{2}, \frac{a}{2}\right\}.$$

The remaining step of the proof is to prove the implication

$$|x-a| < \min\left\{\frac{a^2\epsilon}{2}, \frac{a}{2}\right\} \quad \Rightarrow \quad \left|\frac{1}{x}-\frac{1}{a}\right| < \epsilon.$$

We hope that at this point you can prove this on your own. Write down the details of your reasoning. □

Exercise 4.10. Find each of the following limits. Prove your claims using Definition 4.1.

(a) $\lim_{x\to 3}(2x+1)$ (b) $\lim_{x\to 1}(-3x-7)$ (c) $\lim_{x\to 1}\left(4x^2+3\right)$

(d) $\lim_{x\to 2}\frac{x}{x-1}$ (e) $\lim_{x\to 3}\frac{x^2-x+2}{x+1}$ (f) $\lim_{x\to 0}x^{1/3}$

(g) $\lim_{x\to 0}\left(\frac{1}{|x|}\right)^{3/\ln|x|}$ (h) $\lim_{x\to 0}\tan x$ (i) $\lim_{x\to 0}\frac{1}{\cos x}$

(j) $\lim_{x\to 3}\frac{1}{x}$ (k) $\lim_{x\to 1}\frac{1}{x^2+1}$ (l) $\lim_{x\to -2}\frac{x}{x^2+4x+3}$

Exercise 4.11. Let $f(x) = \dfrac{x+1}{x^2-1}$. Does f have a limit at $a=1$? Justify your answer.

Exercise 4.12. Prove that for any $a>0$, $\lim_{x\to a}\sqrt{x}=\sqrt{a}$.

4.3. Infinite limits.

Definition 4.13. A function f has the limit $+\infty$ as x approaches a real number a if the following two conditions are satisfied:

(I) There exists a real number $\delta_0 > 0$ such that $f(x)$ is defined for each x in the set $\big(a-\delta_0,a\big)\cup\big(a,a+\delta_0\big)$.

(II) For each real number $M>0$ there exists a real number $\delta(M)$ such that $0<\delta(M)\leq\delta_0$ and

$$0<|x-a|<\delta(M) \quad \Rightarrow \quad f(x)>M.$$

Definition 4.14. A function f has the limit $-\infty$ as x approaches a real number a if the following two conditions are satisfied:

(I) There exists a real number $\delta_0 > 0$ such that $f(x)$ is defined for each x in the set $\big(a-\delta_0,a\big)\cup\big(a,a+\delta_0\big)$.

(II) For each real number $M<0$ there exists a real number $\delta(M)$ such that $0<\delta(M)\leq\delta_0$ and

$$0<|x-a|<\delta(M) \quad \Rightarrow \quad f(x)<M.$$

EXERCISE 4.15. Find each of the following limits. Prove your claims using the appropriate definition.

(a) $\displaystyle\lim_{x\to 0}\frac{1}{|x|}$ (b) $\displaystyle\lim_{x\to -3}\frac{1}{(x+3)^2}$ (c) $\displaystyle\lim_{x\to 2}\frac{x-3}{x(x-2)^2}$

(d) $\displaystyle\lim_{x\to -1}\frac{x}{(x+1)^4}$ (e) $\displaystyle\lim_{x\to +\infty}\frac{x^2-x+2}{x+1}$ (f) $\displaystyle\lim_{x\to +\infty}\frac{x^2-x}{3-x}$

4.4. One-sided limits.

DEFINITION 4.16. A function f has the limit $L\in\mathbb{R}$ as x approaches a real number a from the left if the following two conditions are satisfied:

(I) There exists a real number $\delta_0>0$ such that $f(x)$ is defined for each x in the set $\bigl(a-\delta_0,a\bigr)$.

(II) For each real number $\epsilon>0$ there exists a real number $\delta(\epsilon)$ such that $0<\delta(\epsilon)\leq\delta_0$ and

$$0<a-x<\delta(\epsilon)\quad\Rightarrow\quad |f(x)-L|<\epsilon.$$

If the conditions (I) and (II) in Definition 4.16 are satisfied we write $\displaystyle\lim_{x\uparrow a}f(x)=L$.

DEFINITION 4.17. A function f has the limit $L\in\mathbb{R}$ as x approaches a real number a from the right if the following two conditions are satisfied:

(I) There exists a real number $\delta_0>0$ such that $f(x)$ is defined for each x in the set $\bigl(a,a+\delta_0\bigr)$.

(II) For each real number $\epsilon>0$ there exists a real number $\delta(\epsilon)$ such that $0<\delta(\epsilon)\leq\delta_0$ and

$$0<x-a<\delta(\epsilon)\quad\Rightarrow\quad |f(x)-L|<\epsilon.$$

If the conditions (I) and (II) in Definition 4.17 are satisfied we write $\displaystyle\lim_{x\downarrow a}f(x)=L$.

DEFINITION 4.18. A function f has the limit $+\infty$ as x approaches a real number a from the left if the following two conditions are satisfied:

(I) There exists a real number $\delta_0>0$ such that $f(x)$ is defined for each x in the set $\bigl(a-\delta_0,a\bigr)$.

(II) For each real number $M>0$ there exists a real number $\delta(M)$ such that $0<\delta(M)\leq\delta_0$ and

$$0<a-x<\delta(M)\quad\Rightarrow\quad f(x)>M.$$

If the conditions (I) and (II) in Definition 4.18 are satisfied we write $\displaystyle\lim_{x\uparrow a}f(x)=+\infty$.

DEFINITION 4.19. A function f has the limit $+\infty$ as x approaches a real number a from the right if the following two conditions are satisfied:

(I) There exists a real number $\delta_0>0$ such that $f(x)$ is defined for each x in the set $\bigl(a,a+\delta_0\bigr)$.

(II) For each real number $M>0$ there exists a real number $\delta(M)$ such that $0<\delta(M)\leq\delta_0$ and

$$0<x-a<\delta(M)\quad\Rightarrow\quad f(x)>M.$$

If the conditions (I) and (II) in Definition 4.19 are satisfied we write $\displaystyle\lim_{x\downarrow a}f(x)=+\infty$.

DEFINITION 4.20. A function f has the limit $-\infty$ as x approaches a real number a from the left if the following two conditions are satisfied:

(I) There exists a real number $\delta_0 > 0$ such that $f(x)$ is defined for each x in the set $\bigl(a-\delta_0, a\bigr)$.

(II) For each real number $M < 0$ there exists a real number $\delta(M)$ such that $0 < \delta(M) \leq \delta_0$ and

$$0 < a - x < \delta(M) \quad \Rightarrow \quad f(x) < M.$$

If the conditions (I) and (II) in Definition 4.20 are satisfied we write $\lim\limits_{x \uparrow a} f(x) = -\infty$.

DEFINITION 4.21. A function f has the limit $-\infty$ as x approaches a real number a from the right if the following two conditions are satisfied:

(I) There exists a real number $\delta_0 > 0$ such that $f(x)$ is defined for each x in the set $\bigl(a, a+\delta_0\bigr)$.

(II) For each real number $M < 0$ there exists a real number $\delta(M)$ such that $0 < \delta(M) \leq \delta_0$ and

$$0 < x - a < \delta(M) \quad \Rightarrow \quad f(x) < M.$$

If the conditions (I) and (II) in Definition 4.21 are satisfied we write $\lim\limits_{x \downarrow a} f(x) = -\infty$.

EXERCISE 4.22. Find each of the following limits. Prove your claims using the appropriate definition.

(a) $\displaystyle \lim_{x \uparrow 5} \frac{3x-15}{\sqrt{x^2-10x+25}}$ (b) $\displaystyle \lim_{x \downarrow 5} \frac{3x-15}{\sqrt{x^2-10x+25}}$ (c) $\displaystyle \lim_{x \uparrow 2} \frac{x-3}{x(x-2)}$

(d) $\displaystyle \lim_{x \downarrow 0} \left(\frac{1}{x} - \frac{1}{x^2}\right)$ (e) $\displaystyle \lim_{x \uparrow 5} \frac{2}{\sqrt{5-x}}$ (f) $\displaystyle \lim_{x \downarrow 5} \frac{6}{5-x}$

(g) $\displaystyle \lim_{x \uparrow 3} \frac{x+3}{x^2-9}$ (h) $\displaystyle \lim_{x \uparrow -3} \frac{x^2}{x^2-9}$ (i) $\displaystyle \lim_{x \downarrow 0} \bigl(x - \sqrt{x}\bigr)$

(j) $\displaystyle \lim_{x \to 3} \frac{x}{(x-3)^2}$ (k) $\displaystyle \lim_{x \downarrow -1} \frac{x^2}{x+1}$ (l) $\displaystyle \lim_{x \to +\infty} \bigl(x - \sqrt{x}\bigr)$

5. New limits from old

5.1. Squeeze theorems. In this section and in Section 5.3 we establish general properties of limits which are based on the formal definition of limit. These properties are stated as theorems.

Establishing theorems of this kind involves a major step forward in sophistication. Up to this point we have been trying to show that limits exist directly from the definition. Now for the first time we are going to **assume** that some limit exists (I refer to this in class as a *green* limit.) and try to make use of this information to establish the existence of some other limit (I refer to this in class as a *red* limit.). Remember that to establish the existence of a limit, we had to come up with a procedure for finding $\delta(\epsilon)$ that will work for any $\epsilon > 0$ that is given. If we assume the existence of a limit, then we are assuming the existence of such a procedure, though we may not know explicitly what it is. I refer to this as a *green* $\delta(\epsilon)$. It is this procedure we will need to use in order to construct a new procedure for the limit whose existence we are trying to establish. I refer to this as a *red* $\delta(\epsilon)$.

We start by considering squeeze theorems that resemble the role of BIN in previous sections. The following theorem is the Sandwich Squeeze Theorem.

Theorem 5.1. *Let f, g and h be given functions and let a and L be real numbers. Suppose that the following three conditions are satisfied.*

(1) $\lim_{x\to a} f(x) = L$.
(2) $\lim_{x\to a} h(x) = L$.
(3) *There exists $\eta_0 > 0$ such that f, g and h are defined on $\big(a-\eta_0, a\big) \cup \big(a, a+\eta_0\big)$ and*

$$f(x) \le g(x) \le h(x) \quad \text{for all} \quad x \in \big(a-\eta_0, a\big) \cup \big(a, a+\eta_0\big).$$

Then

$$\lim_{x\to a} g(x) = L.$$

Proof. Here we have three functions and three definitions of limits, one for each function. Therefore we have to deal with three δ-s. We will give them appropriate names that will distinguish them from each other. Let us name them δ_f, δ_g and δ_h.

In the theorem it is assumed that $\lim_{x\to a} f(x) = L$. This means that we are given the fact that for each $\epsilon > 0$ there exists $\delta_f(\epsilon) > 0$ (that is, we are given a function $\delta_f(\epsilon)$) such that

$$0 < |x-a| < \delta_f(\epsilon) \quad \Rightarrow \quad |f(x) - L| < \epsilon. \tag{5.1}$$

In class I refer to these as a green $\delta_f(\cdot)$ and a green implication.

Since the theorem assumes that $\lim_{x\to a} h(x) = L$, we are also given that for each $\epsilon > 0$ there exists $\delta_h(\epsilon) > 0$ such that

$$0 < |x-a| < \delta_h(\epsilon) \quad \Rightarrow \quad |h(x) - L| < \epsilon. \tag{5.2}$$

Again we refer to these as a green $\delta_h(\cdot)$ and a green implication.

We need to prove that $\lim_{x\to a} g(x) = L$. Therefore, following the definition of limit, we have to show that the following conditions are satisfied:

(I) There exists a real number $\delta_{0,g} > 0$ such that $g(x)$ is defined for each x in the set $\big(a-\delta_{0,g}, a\big) \cup \big(a, a+\delta_{0,g}\big)$.

(II) For each real number $\epsilon > 0$ there exists a real number $\delta_g(\epsilon)$ such that $0 < \delta_g(\epsilon) \leq \delta_{0,g}$ and such that

$$0 < |x - a| < \delta_g(\epsilon) \quad \Rightarrow \quad |g(x) - L| < \epsilon. \tag{5.3}$$

Since we have to produce $\delta_{0,g}, \delta_g(\epsilon)$ and we have to prove the last implication, all of these objects are red.

Notice that η_0 in the theorem is green.

The objective here is to use the green objects to produce the red objects. We will do that next. We put:

(I) $\delta_{0,g} = \eta_0$. By the assumption of the theorem $g(x)$ is defined for each x in the set $\bigl(a - \eta_0, a\bigr) \cup \bigl(a, a + \eta_0\bigr)$.

(II) For each real number $\epsilon > 0$, put

$$\delta_g(\epsilon) = \min\bigl\{\delta_f(\epsilon), \delta_h(\epsilon), \eta_0\bigr\}.$$

This is a beautiful expression since the red object is expressed in terms of the green objects.

It remains to prove the red implication (5.3) using the green implications and the assumptions of the theorem.

To prove (5.3), $\boxed{\text{assume that } 0 < |x - a| < \delta_g(\epsilon).}$ Then, clearly, $0 < |x - a| < \eta_0$. This is telling me that $x \neq a$ and that x is no further than η_0 from a. Consequently, $x \in \bigl(a - \eta_0, a\bigr) \cup \bigl(a, a + \eta_0\bigr)$. Therefore, by the assumption of the theorem

$$f(x) \leq g(x) \leq h(x).$$

Subtracting L from each term in this inequality, we conclude that

$$f(x) - L \leq g(x) - L \leq h(x) - L.$$

Using the property of the absolute value that $-|u| \leq u \leq |u|$ for each real number u, we conclude that

$$-|f(x) - L| \leq f(x) - L \leq g(x) - L \leq h(x) - L \leq |h(x) - L|. \tag{5.4}$$

From the assumption $0 < |x - a| < \delta_g(\epsilon)$, we conclude that $0 < |x - a| < \delta_f(\epsilon)$. By the green implication (5.1), this implies that $|f(x) - L| < \epsilon$ and therefore

$$-\epsilon < -|f(x) - L|. \tag{5.5}$$

From the assumption $0 < |x - a| < \delta_g(\epsilon)$, we conclude that $0 < |x - a| < \delta_h(\epsilon)$. By the green implication (5.2), this implies that

$$|h(x) - L| < \epsilon. \tag{5.6}$$

Putting together the inequalities (5.4), (5.5) and (5.6), we conclude that

$$-\epsilon < g(x) - L < \epsilon. \tag{5.7}$$

The inequalities in (5.7) are equivalent to

$$|g(x) - L| < \epsilon.$$

This proves that $0 < |x - a| < \delta_g(\epsilon)$ implies $|g(x) - L| < \epsilon$ and this is exactly the red implication (5.3). This completes the proof. □

The following theorem is the Scissors Squeeze Theorem.

THEOREM 5.2. *Let f, g and h be given functions and let $a \in \mathbb{R}$ and $L \in \mathbb{R}$. Assume that*

(1) $\lim_{x\to a} f(x) = L$.
(2) $\lim_{x\to a} h(x) = L$.
(3) *There exists* $\eta_0 > 0$ *such that* f, g *and* h *are defined on* $\big(a-\eta_0, a\big) \cup \big(a, a+\eta_0\big)$ *and*

$$f(x) \le g(x) \le h(x) \quad \textit{for all} \quad x \in \big(a-\eta_0, a\big),$$

and

$$h(x) \le g(x) \le f(x) \quad \textit{for all} \quad x \in \big(a, a+\eta_0\big).$$

Then

$$\lim_{x\to a} g(x) = L.$$

5.2. Examples for squeeze theorems. Figure 19 and the numbers that *you can see on it* are essential for getting squeezes for limits involving trigonometric functions. The table to the left of Figure 19 shows the numbers that you should be able to identify on the picture.

Geometric object	Associated number
Circular arc from C to B	u
Line segment $\overline{OA}$	$\cos u$
Line segment $\overline{AB}$	$\sin u$
Line segment $\overline{AC}$	$1-\cos u$
Line segment $\overline{CB}$	You calculate
Line segment $\overline{CD}$	$\tan u$
Line segment $\overline{OB}$	1
Line segment $\overline{OC}$	1

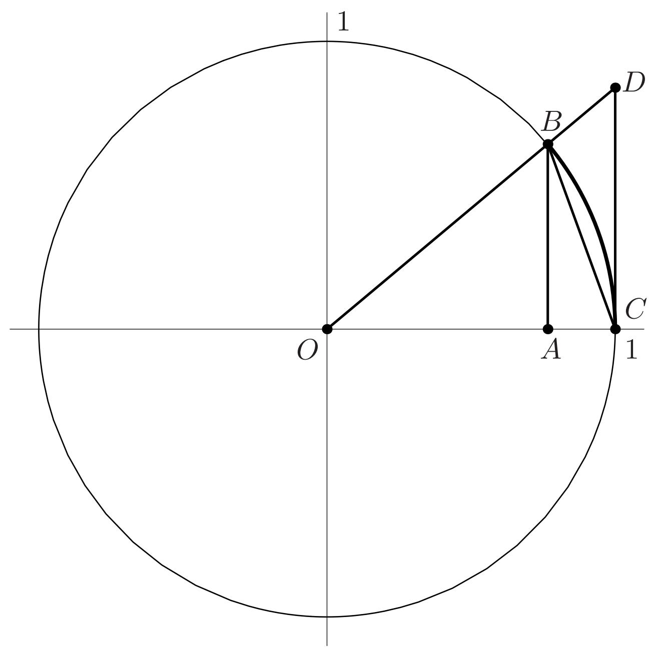

FIG. 19. The unit circle

EXAMPLE 5.3. Prove that $\lim_{x\to 0} \cos x = 1$.

SOLUTION. Set $\eta_0 = \dfrac{\pi}{3}$. Consider positive u. Look at the picture above. The triangle $\triangle ACB$ is a right triangle. Therefore its hypothenuse, the line segment $\overline{CB}$, is longer than its side $\overline{AC}$ which equals to $1-\cos u$. Thus

$$1-\cos u = \overline{AC} \le \overline{CB}. \tag{5.8}$$

The line segment $\overline{CB}$ is a segment of a straight line, therefore it is shorter than any other curve joining C and B. In particular it is shorter than the circular arc joining the points C and B. The length of this circular arc is u. Thus

$$\overline{CB} \leq \text{ Length of the Circular Arc from } C \text{ to } B \;(= \; u\;). \tag{5.9}$$

Putting together the inequalities (5.8) and (5.9), we conclude that

$$1 - \cos u \leq u \quad \text{for all} \quad 0 < u < \frac{\pi}{3}. \tag{5.10}$$

Since the length $\overline{OA} = \cos u$ is smaller than 1, from (5.10) we conclude that

$$0 \leq 1 - \cos u \leq u \quad \text{for all} \quad 0 < u < \frac{\pi}{3}\;,$$

or, equivalently,

$$1 - u \leq \cos u \leq 1 \quad \text{for all} \quad 0 < u < \frac{\pi}{3}\;,$$

Now we substitute $u = |x|$ and use the fact that $\cos |x| = \cos x$ and (5.2) becomes

$$1 - |x| \leq \cos x \leq 1 \quad \text{for all} \quad -\frac{\pi}{3} < x < \frac{\pi}{3}.$$

This is a sandwich squeeze for $\cos x$. It is easy to prove that $\lim_{x\to 0} 1 = 1$ and $\lim_{x\to 0}\big(1 - |x|\big) = 1$. (Please prove this using the definition!) Now the Sandwich Squeeze Theorem implies that $\lim_{x\to 0} \cos x = 1$. □

Example 5.4. Prove that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$.

Solution. To get a sandwich squeeze for this problem consider the following three areas on the picture above.

Area 1 The triangle $\triangle OCB$.

Area 2 The segment of the unit disc bounded by the line segments $\overline{OC}$ and $\overline{OB}$ and the circular arc segment joining points C and B.

Area 3 The triangle $\triangle OCD$.

The picture tells clearly the inequality between these areas. Write that inequality. Calculate each area in terms of the numbers that appear in the table above. This will lead to the inequality, which when simplified gives

$$\cos u \leq \frac{\sin u}{u} \leq 1 \quad \text{for all} \quad 0 < u < \frac{\pi}{3}. \tag{5.11}$$

Using the same idea as in the previous example, the inequality (5.11) leads to

$$\cos x \leq \frac{\sin x}{x} \leq 1 \quad \text{for all} \quad x \in \Big(-\frac{\pi}{3}, 0\Big) \cup \Big(0, \frac{\pi}{3}\Big)\,. \tag{5.12}$$

The inequality (5.12) is exactly what we need in the Sandwich Squeeze Theorem. Please fill in all the details of the rest of the proof. □

Example 5.5. Prove that $\lim_{x\to 0} \dfrac{1 - \cos x}{x^2} = \dfrac{1}{2}$.

Solution. To establish squeeze inequlaities consider three lengths:

Length 1 The line segment $\overline{AB}$.

Length 2 The line segment $\overline{CB}$.

Length 3 The length of a circular arc joining the points C and B.

The picture tells clearly the inequalities between these three lengths. Write these inequalities. Calculate each length in terms of the numbers that appear in the table above. This will lead to the inequalities, which, when simplified, give

$$\frac{1}{2}\left(\frac{\sin u}{u}\right)^2 \leq \frac{1-\cos u}{u^2} \leq \frac{1}{2} \quad \text{for all} \quad 0 < u < \frac{\pi}{3}. \tag{5.13}$$

From the inequality (5.13) and one inequality established in a previous example you can get an "easy" sandwich squeeze. Please fill in all the details of the rest of the proof. □

EXAMPLE 5.6. Prove that $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = 1$.

SOLUTION. The idea is to use the definition of $\ln$ as an integral and work with areas to get squeeze inequalities. □

5.3. Algebra of limits. A nickname that I gave to a function which has a limit L when x approaches a is: f is *constantish* L near a. If we are dealing with constant functions $f(x) = L$ and $g(x) = K$, then clearly the sum $f + g$ of these two functions is a constant function equal to $L+K$. The same is true for the product fg which is the constant function equal to LK. Another question is whether we can talk about the reciprocal $1/f$. If $L \neq 0$, then the reciprocal of f is defined and it equals $1/L$. In this section we will prove that all these properties hold for constantish functions.

THEOREM 5.7. *Let f, g, and h, be functions with domain and range in $\mathbb{R}$. Let a, K and L be real numbers. Assume that*

(1) $\lim\limits_{x\to a} f(x) = K$,
(2) $\lim\limits_{x\to a} g(x) = L$.

Then the following statements hold.

(A) *If $h = f + g$, then $\lim\limits_{x\to a} h(x) = K + L$.*
(B) *If $h = fg$, then $\lim\limits_{x\to a} h(x) = KL$.*
(C) *If $L \neq 0$ and $h = \dfrac{1}{g}$, then $\lim\limits_{x\to a} h(x) = \dfrac{1}{L}$.*
(D) *If $L \neq 0$ and $h = \dfrac{f}{g}$, then $\lim\limits_{x\to a} h(x) = \dfrac{K}{L}$.*

PROOF. The assumption $\lim\limits_{x\to a} f(x) = K$ implies that

green(I-f) There exists (green!) $\delta_{0,f} > 0$ such that $f(x)$ is defined for all x in $\big(a - \delta_{0,f}, a\big) \cup \big(a, a + \delta_{0,f}\big)$;

green(II-f) For each $\epsilon > 0$ there exists (green!) $\delta_f(\epsilon)$ such that $0 < \delta_f(\epsilon) \leq \delta_{0,f}$ and such that

$$0 < |x - a| < \delta_f(\epsilon) \quad \Rightarrow \quad |f(x) - K| < \epsilon. \tag{5.14}$$

The assumption $\lim\limits_{x\to a} g(x) = L$ implies that

green(I-g) There exists (green!) $\delta_{0,g} > 0$ such that $g(x)$ is defined for all x in $\big(a-\delta_{0,g}, a\big) \cup \big(a, a + \delta_{0,g}\big)$;

green(II-g) For each $\epsilon > 0$ there exists (green!) $\delta_g(\epsilon)$ such that $0 < \delta_g(\epsilon) \leq \delta_{0,g}$ and such that

$$0 < |x - a| < \delta_g(\epsilon) \quad \Rightarrow \quad |g(x) - L| < \epsilon. \tag{5.15}$$

Proof of the statement (A). Remember that $h(x) = f(x) + g(x)$ here. First we list what is red in this proof.

red(I-h) There exists (red!) $\delta_{0,h} > 0$ such that $h(x)$ is defined for all x in $\bigl(a - \delta_{0,h}, a\bigr) \cup \bigl(a, a + \delta_{0,h}\bigr)$;

red(II-h) For each $\epsilon > 0$ there exists (red!) $\delta_h(\epsilon)$ such that $0 < \delta_h(\epsilon) \leq \delta_{0,h}$ and such that

$$0 < |x - a| < \delta_h(\epsilon) \quad \Rightarrow \quad |h(x) - (K + L)| < \epsilon. \tag{5.16}$$

I will not elaborate here how I got the idea for $\delta_{0,h}$ and $\delta_h(\epsilon)$, I will just give formulas and convince you that my choice is a correct one. The idea for the formulas comes from the boxed paragraph on page 32. I invite you to enjoy the separation of colors in the following formulas.

Let $\epsilon > 0$ be given. Put

$$\delta_{0,h} := \min\left\{\delta_{0,f}, \delta_{0,g}\right\}$$
$$\delta_h(\epsilon) := \min\left\{\delta_f\left(\frac{\epsilon}{2}\right), \delta_g\left(\frac{\epsilon}{2}\right)\right\}$$

Now we have to prove that $h(x)$ is defined for each $x \in \bigl(a - \delta_{0,h}, a\bigr) \cup \bigl(a, a + \delta_{0,h}\bigr)$. Assume that $x \in \bigl(a - \delta_{0,h}, a\bigr) \cup \bigl(a, a + \delta_{0,h}\bigr)$. Then

$$0 < |x - a| < \delta_{0,h} \leq \min\left\{\delta_{0,f}, \delta_{0,g}\right\}. \tag{5.17}$$

It follows from (5.17) that

$$0 < |x - a| < \delta_{0,f},$$

and therefore $x \in \bigl(a - \delta_{0,f}, a\bigr) \cup \bigl(a, a + \delta_{0,f}\bigr)$. Thus $f(x)$ is defined. It also follows from (5.17) that

$$0 < |x - a| < \delta_{0,g},$$

and therefore $x \in \bigl(a - \delta_{0,g}, a\bigr) \cup \bigl(a, a + \delta_{0,g}\bigr)$. Thus $g(x)$ is defined. Therefore $h(x) = f(x) + g(x)$ is defined for each $x \in \bigl(a - \delta_{0,h}, a\bigr) \cup \bigl(a, a + \delta_{0,h}\bigr)$.

Now we will prove the red implication (5.16). Assume

$$0 < |x - a| < \delta_h(\epsilon) = \min\left\{\delta_f\left(\frac{\epsilon}{2}\right), \delta_g\left(\frac{\epsilon}{2}\right)\right\}. \tag{5.18}$$

Then

$$0 < |x - a| < \delta_f\left(\frac{\epsilon}{2}\right). \tag{5.19}$$

The inequality (5.19) and the implication (5.14) allow me to conclude that

$$|f(x) - K| < \frac{\epsilon}{2}. \tag{5.20}$$

It follows from (5.18) that

$$0 < |x - a| < \delta_g\left(\frac{\epsilon}{2}\right). \tag{5.21}$$

The inequality (5.21) and the implication (5.15) allow me to conclude that

$$|g(x) - L| < \frac{\epsilon}{2}. \tag{5.22}$$

Now we remember that the absolute value has the property that $|u+v| \leq |u|+|v|$. We will apply this to the expression

$$|h(x)-(K+L)| = |f(x)+g(x)-K-L| = |\underbrace{(f(x)-K)}_{u}+\underbrace{(g(x)-L)}_{v}|$$

to get

$$|h(x)-(K+L)| \leq |f(x)-K|+|g(x)-L|. \tag{5.23}$$

This inequality plays a role of a BIN in this abstract proof. It has an unfriendly object on the left and all friendly objects on the right.

The inequalities (5.20), (5.22) and (5.23) imply that

$$|h(x)-(K+L)| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon. \tag{5.24}$$

Reviewing my reasoning above you should be convinced that based on the assumption (5.18) we proved the inequality (5.24). This is exactly the implication (5.16). This completes the proof of the statement (A).

Proof of the statement (B). Remember that $h(x) = f(x)g(x)$ here. We first list what is red in this proof.

red(I-h) There exists (red!) $\delta_{0,h} > 0$ such that $h(x)$ is defined for all x in $\bigl(a-\delta_{0,h}, a\bigr) \cup \bigl(a, a+\delta_{0,h}\bigr)$;

red(II-h) For each $\epsilon > 0$ there exists (red!) $\delta_h(\epsilon)$ such that $0 < \delta_h(\epsilon) \leq \delta_{0,h}$ and such that

$$0 < |x-a| < \delta_h(\epsilon) \quad \Rightarrow \quad |h(x)-KL| < \epsilon. \tag{5.25}$$

I will not elaborate how I got the idea for $\delta_{0,h}$ and $\delta_h(\epsilon)$, I will just give formulas and convince you that my choice is a correct one. The idea for the formulas comes from the boxed paragraph on page 33. Again, I invite you to enjoy the separation of colors in the following formulas.

Let $\epsilon > 0$ be given. Put

$$\delta_{0,h} := \min\left\{\delta_{0,f}, \delta_g(1)\right\}$$

$$\delta_h(\epsilon) := \min\left\{\delta_f\left(\frac{\epsilon}{2(|L|+1)}\right), \delta_g\left(\frac{\epsilon}{2(|K|+1)}\right)\right\}.$$

Now we have to prove that $h(x)$ is defined for each $x \in \bigl(a-\delta_{0,h}, a\bigr) \cup \bigl(a, a+\delta_{0,h}\bigr)$. Assume that $x \in \bigl(a-\delta_{0,h}, a\bigr) \cup \bigl(a, a+\delta_{0,h}\bigr)$. Then

$$0 < |x-a| < \delta_{0,h} \leq \min\left\{\delta_{0,f}, \delta_g(1)\right\}. \tag{5.26}$$

It follows from (5.26) that

$$0 < |x-a| < \delta_{0,f}\ ,$$

and therefore $x \in \bigl(a-\delta_{0,f}, a\bigr) \cup \bigl(a, a+\delta_{0,f}\bigr)$. Thus $f(x)$ is defined. It also follows from (5.26) that

$$0 < |x-a| < \delta_g(1). \tag{5.27}$$

Since by the assumption (II-g) we know that $\delta_g(1) \leq \delta_{0,g}$, the inequality (5.27) implies that

$$0 < |x-a| < \delta_{0,g}.$$

Therefore $x \in \bigl(a-\delta_{0,g}, a\bigr) \cup \bigl(a, a+\delta_{0,g}\bigr)$. Thus $g(x)$ is defined. Therefore $h(x) = f(x)g(x)$ is defined for each $x \in \bigl(a-\delta_{0,h}, a\bigr) \cup \bigl(a, a+\delta_{0,h}\bigr)$.

At this point we will prove another consequence of the inequality (5.27). This inequality and the implication (5.15) allow me to conclude that

$$|g(x) - L| < 1.$$

Therefore

$$-1 < g(x) - L < 1 \ ,$$

or, equivalently

$$-1 + L < g(x) < L + 1.$$

Multiplying the last inequality by -1, we conclude that

$$-1 - L < -g(x) < -L + 1.$$

From the last two inequalities we conclude that $\max\{g(x), -g(x)\} < \max\{L+1, -L+1\} = \max\{L, -L\} + 1$. Thus

$$|g(x)| < |L| + 1. \tag{5.28}$$

Now we will prove the red implication (5.25). Assume

$$0 < |x - a| < \delta_h(\epsilon) = \min\left\{\delta_f\left(\frac{\epsilon}{2(|L|+1)}\right), \delta_g\left(\frac{\epsilon}{2(|K|+1)}\right)\right\}. \tag{5.29}$$

Then

$$0 < |x - a| < \delta_f\!\left(\frac{\epsilon}{2(|L|+1)}\right). \tag{5.30}$$

The inequality (5.30) and the implication (5.14) allow me to conclude that

$$|f(x) - K| < \frac{\epsilon}{2(|L|+1)}. \tag{5.31}$$

It follows from (5.29) that

$$0 < |x - a| < \delta_g\!\left(\frac{\epsilon}{2(|K|+1)}\right). \tag{5.32}$$

The inequality (5.32) and the implication (5.15) allow me to conclude that

$$|g(x) - L| < \frac{\epsilon}{2(|K|+1)}. \tag{5.33}$$

Now we remember that the absolute value has the property that $|u + v| \leq |u| + |v|$ and that $|uv| = |u||v|$. we will apply these properties to the expression

$$\begin{aligned}|h(x) - KL| = \big|f(x)g(x) - KL\big| &= \big|\underbrace{\big(f(x)g(x) - Kg(x)\big)}_{u} + \underbrace{\big(Kg(x) - KL\big)}_{v}\big| \\ &\leq \big|f(x)g(x) - Kg(x)\big)\big| + \big|Kg(x) - KL\big| \\ &\leq \big|g(x)\big|\,\big|f(x) - K\big| + |K|\,\big|g(x) - L\big|.\end{aligned}$$

Summarizing

$$\big|h(x) - KL\big| \leq \big|g(x)\big|\,\big|f(x) - K\big| + |K|\,\big|g(x) - L\big|. \tag{5.34}$$

The inequalities (5.28) and (5.34) imply that

$$\big|h(x) - KL\big| \leq \big(|L| + 1\big)\,\big|f(x) - K\big| + |K|\,\big|g(x) - L\big|. \tag{5.35}$$

This inequality plays a role of a BIN in this abstract proof. It has an unfriendly object on the left and all friendly objects on the right.

The inequalities (5.31), (5.33) and (5.35) imply that

$$|h(x) - LK| \le \big(|L|+1\big)\,\frac{\epsilon}{2(|L|+1)} + |K|\,\frac{\epsilon}{2(|K|+1)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{5.36}$$

I hope that my reasoning above convinces you that the assumption (5.29) implies the inequality (5.36). This is exactly the implication (5.25). This completes the proof of the part (B).

Proof of the statement (C). Here we assume that $L \neq 0$ and $h(x) = \dfrac{1}{g(x)}$. Next we list what is red in this proof.

red(I-h) There exists (red!) $\delta_{0,h} > 0$ such that $h(x)$ is defined for all x in $\big(a-\delta_{0,h}, a\big) \cup \big(a, a+\delta_{0,h}\big)$;

red(II-h) For each $\epsilon > 0$ there exists (red!) $\delta_h(\epsilon)$ such that $0 < \delta_h(\epsilon) \le \delta_{0,h}$ and such that

$$0 < |x-a| < \delta_h(\epsilon) \quad \Rightarrow \quad \left|\frac{1}{g(x)} - \frac{1}{L}\right| < \epsilon. \tag{5.37}$$

I will not elaborate how I got the idea for $\delta_{0,h}$ and $\delta_h(\epsilon)$, I will just give formulas and convince you that my choice is a correct one. The idea for the formulas comes from the boxed paragraph on page 35. Again, I invite you to enjoy the separation of colors in the following formulas.

Let $\epsilon > 0$ be given. Remember that it is assumed that $|L| > 0$. Put

$$\delta_{0,h} := \delta_g\!\left(\frac{|L|}{2}\right)$$

$$\delta_h(\epsilon) := \min\left\{\delta_g\!\left(\frac{\epsilon L^2}{2}\right), \delta_g\!\left(\frac{|L|}{2}\right)\right\}.$$

Now we have to prove that $h(x)$ is defined for each $x \in \big(a-\delta_{0,h}, a\big) \cup \big(a, a+\delta_{0,h}\big)$. Assume that $x \in \big(a-\delta_{0,h}, a\big) \cup \big(a, a+\delta_{0,h}\big)$. Then

$$0 < |x-a| < \delta_{0,h} = \delta_g\!\left(\frac{|L|}{2}\right).$$

This inequality and the implication (5.15) allow me to conclude that

$$|g(x) - L| < \frac{|L|}{2}.$$

Therefore

$$-\frac{|L|}{2} < g(x) - L < \frac{|L|}{2}\ ,$$

or, equivalently

$$-\frac{|L|}{2} + L < g(x) < L + \frac{|L|}{2}.$$

Multiplying the last inequality by -1, we conclude that

$$-L - \frac{|L|}{2} < -g(x) < \frac{|L|}{2} - L.$$

From the last two displayed relationships we conclude that

$$\max\{g(x), -g(x)\} > \max\left\{L - \frac{|L|}{2}, -L - \frac{|L|}{2}\right\} = \max\{L, -L\} - \frac{|L|}{2}.$$

Thus

$$|g(x)| > |L| - \frac{|L|}{2} = \frac{|L|}{2} > 0. \tag{5.38}$$

Consequently, $g(x) \neq 0$. Therefore, $h(x) = \dfrac{1}{g(x)}$ is defined for all $x \in \big(a - \delta_{0,h}, a\big) \cup \big(a, a + \delta_{0,h}\big)$.

Now we will prove the red implication (5.37). Assume

$$0 < |x-a| < \delta_h(\epsilon) = \min\left\{ \delta_g\!\left(\frac{\epsilon L^2}{2}\right), \delta_g\!\left(\frac{|L|}{2}\right) \right\}. \tag{5.39}$$

Then

$$0 < |x-a| < \delta_g\!\left(\frac{\epsilon L^2}{2}\right). \tag{5.40}$$

The inequality (5.40) and the implication (5.15) allow me to conclude that

$$|g(x) - L| < \frac{\epsilon L^2}{2}. \tag{5.41}$$

It also follows from (5.39) that

$$0 < |x-a| < \delta_g\!\left(\frac{|L|}{2}\right).$$

We already proved that this inequality implies (5.38). Therefore

$$\frac{1}{|g(x)|} < \frac{2}{|L|}. \tag{5.42}$$

This inequality is used at the last step in the sequence of inequalities below. In some sense this is an abstract version of a "pizza-party" play.

Using our standard tools, algebra, properties of the absolute value and the inequality (5.42) we get

$$\begin{aligned} \left| h(x) - \frac{1}{L} \right| &= \left| \frac{1}{g(x)} - \frac{1}{L} \right| = \left| \frac{L - g(x)}{g(x)L} \right| = \frac{|L - g(x)|}{|g(x)| \; |L|} \\ &= \frac{|g(x) - L|}{|g(x)| \; |L|} \leq \frac{1}{|g(x)|} \, \frac{|g(x) - L|}{|L|} \leq \frac{2}{|L|} \, \frac{|g(x) - L|}{|L|}. \end{aligned}$$

Summarizing

$$\left| \frac{1}{g(x)} - \frac{1}{L} \right| \leq \frac{2}{L^2} \; |g(x) - L| \,. \tag{5.43}$$

This inequality plays a role of a BIN in this abstract proof. It has an unfriendly object on the left and all friendly objects on the right.

The inequalities (5.41) and (5.43) imply that

$$\left| \frac{1}{g(x)} - \frac{1}{L} \right| \leq \frac{2}{L^2} \, \frac{\epsilon L^2}{2} = \epsilon. \tag{5.44}$$

I hope that the reasoning above convinces you that the assumption (5.39) implies the inequality (5.44). This is exactly the implication (5.37). This completes the proof of the part (C).

Proof of the statement (D). Here we assume that $L \neq 0$ and $h(x) = \dfrac{f(x)}{g(x)}$. We can prove the statement (D) by using the universal power of the statements (B) and (C). First define the functions $g_1(x) = \dfrac{1}{g(x)}$. Then, by the statement (C) we know

$$\lim_{x \to a} g_1(x) = \frac{1}{L}. \tag{5.45}$$

Clearly, $h(x) = f(x)g_1(x)$. Now we can apply the statement (B) to this function h. Taking into account (5.45) the statement (B) implies

$$\lim_{x \to a} h(x) = K\,\frac{1}{L} = \frac{K}{L}.$$

This completes the proof of the statement (D). The theorem is proved. □

Exercise 5.8. Use the algebra of limits to give much simpler proofs for most of the limits in the previous exercises and examples.

6. Continuous functions

6.1. The definition and examples. All this work about limits will now pay off since we will be able to give mathematically rigorous definition of a continuous function.

DEFINITION 6.1. Let f be a real valued function of a real variable and let a be a real number. The function f is continuous at a if the following two conditions are satisfied:

(i) The function f is defined at a, that is $f(a)$ is defined.
(ii) $\lim_{x\to a} f(x) = f(a)$.

To understand Definition 6.1 the reader has to understand the concept of limit. Sometimes it is useful to state the definition of continuity directly, without appealing to the concept of limit.

DEFINITION 6.2. Let f be a real valued function of a real variable and let a be a real number. The function f is continuous at a if the following two conditions are satisfied:

(I) There exists a $\delta_0 > 0$ such that $f(x)$ is defined for all $x \in (a - \delta_0, a + \delta_0)$.
(II) For each $\epsilon > 0$ there exists $\delta(\epsilon)$ such that $0 < \delta(\epsilon) \leq \delta_0$ and such that

$$|x - a| < \delta(\epsilon) \quad \Rightarrow \quad |f(x) - f(a)| < \epsilon.$$

Definition 6.2 is called ϵ-δ definition of continuity.

DEFINITION 6.3. Let I be an interval in $\mathbb{R}$. A function f is *continuous on* I if it is continuous at each point in I.

EXAMPLE 6.4. Let c be a real number and define $f(x) = c$ for all $x \in \mathbb{R}$. Use Definition 6.2 to prove that f is continuous at an arbitrary real number a.

EXAMPLE 6.5. Let $f(x) = x$ for all $x \in \mathbb{R}$. Use Definition 6.2 to prove that f is continuous at an arbitrary real number a.

EXAMPLE 6.6. Use ϵ-δ definition of continuity, that is Definition 6.2, to prove that the function $f(x) = 1/x$ is continuous on the interval $(0, +\infty)$.

SOLUTION. Let $a \in (0, +\infty)$, that is let a be an arbitrary positive number. Chose $\delta_0 = a/2$. Since $a > 0$, we conclude that $a/2 > 0$ and $f(x) = 1/x$ is defined for all $x \in \bigl(a/2, 3a/2\bigr)$.

Let $\epsilon > 0$ be arbitrary. Now we have to solve

$$\left|\frac{1}{x} - \frac{1}{a}\right| < \epsilon \qquad \text{for} \qquad |x - a|.$$

First simplify the expression, using the fact that $x > 0$ and $a > 0$ and rules for the absolute value:

$$\left|\frac{1}{x} - \frac{1}{a}\right| = \left|\frac{a - x}{x\,a}\right| = \frac{|a - x|}{|x|\,|a|} = \frac{|x - a|}{x\,a}.$$

To get a larger expression which will be easy to solve we replace x in the denominator by the smallest possible value for x. That value is $a - a/2 = a/2$. This gives me my BIN:

$$\left|\frac{1}{x} - \frac{1}{a}\right| = \frac{|x - a|}{x\,a} \leq \frac{|x - a|}{\frac{a}{2}\,a} = 2\,\frac{|x - a|}{a^2}.$$

Thus my BIN is $\displaystyle \left|\frac{1}{x} - \frac{1}{a}\right| \leq 2\,\frac{|x - a|}{a^2}$ valid for all $x \in \bigl(a/2, 3a/2\bigr)$.

Solving the inequality $2\,\dfrac{|x-a|}{a^2} < \epsilon$ for $|x-a|$ is easy. The solution is $|x-a| < a^2\,\epsilon/2$. Now we define

$$\delta(\epsilon) = \min\left\{\frac{a^2\,\epsilon}{2}, \frac{a}{2}\right\}.$$

To finish the proof, it remains to prove the implication

$$|x-a| < \delta(\epsilon) \quad \Rightarrow \quad \left|\frac{1}{x} - \frac{1}{a}\right| < \epsilon.$$

This should be easy, using the BIN. □

EXAMPLE 6.7. Use ϵ-δ definition of continuity, that is Definition 6.2, to prove that the function $x \mapsto \sqrt{x}$ is continuous on the interval $(0, +\infty)$.

SOLUTION. Let $a \in (0, +\infty)$. Chose $\delta_0 = \dfrac{a}{2}$. Since $a > 0$, as before we conclude that $\dfrac{a}{2} > 0$ and the function $x \mapsto \sqrt{x}$ is defined for all $x \in (a/2, 3a/2)$.

Let $\epsilon > 0$ be arbitrary. Now we have to solve

$$\left|\sqrt{x} - \sqrt{a}\right| < \epsilon \quad \text{for} \quad |x-a|.$$

First simplify algebraically the expression, using the fact that $x > 0$ and $a > 0$ and rules for the absolute value.

$$\begin{aligned}\left|\sqrt{x} - \sqrt{a}\right| &= \left|\left(\sqrt{x} - \sqrt{a}\right)\,\frac{1}{1}\right| = \left|\left(\sqrt{x} - \sqrt{a}\right)\,\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}\right| = \left|\frac{x-a}{\sqrt{x}+\sqrt{a}}\right| \\ &= \frac{|x-a|}{|\sqrt{x}+\sqrt{a}|} = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} \leq \frac{|x-a|}{\sqrt{a}}\end{aligned}$$

Thus my BIN is: $\left|\sqrt{x} - \sqrt{a}\right| \leq \dfrac{|x-a|}{\sqrt{a}}$, valid for $x > 0$.

Solving $\dfrac{|x-a|}{\sqrt{a}} < \epsilon$ for $|x-a|$ is easy: The solution is $|x-a| < \sqrt{a}\,\epsilon$. Now we define

$$\delta(\epsilon) = \min\left\{\sqrt{a}\,\epsilon, \frac{a}{2}\right\}.$$

It remains to prove the implication $|x-a| < \min\left\{\sqrt{a}\,\epsilon, \dfrac{a}{2}\right\} \;\Rightarrow\; |\sqrt{x} - \sqrt{a}| < \epsilon$. This should be easy, using the BIN. □

EXAMPLE 6.8. Let $f(x) = \dfrac{1}{x^2+1}$ for all $x \in \mathbb{R}$. Use ϵ-δ definition to prove that f is continuous at an arbitrary $a \in \mathbb{R}$.

EXAMPLE 6.9. Let a, b, c be any real numbers. Let $f(x) = ax^2 + bx + c$ for all $x \in \mathbb{R}$. Let v be an arbitrary real number. Prove that f is continuous at v.

EXAMPLE 6.10. Let $f(x) = \sin x$ for all $x \in \mathbb{R}$. Prove that f is continuous at an arbitrary real number a.

EXAMPLE 6.11. Let $f(x) = \cos x$ for all $x \in \mathbb{R}$. Prove that f is continuous at an arbitrary real number a.

HINT FOR Exercises 6.10 and 6.11. Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ be two points in the xy-plane. Then the length of the line segment $\overline{AB}$ is given by

$$\overline{AB} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Consequently

$$|x_1 - x_2| \le \overline{AB} \qquad \text{and} \qquad |y_1 - y_2| \le \overline{AB}.$$

Let u and v be real numbers and set $A = (\cos u, \sin u)$, $B = (\cos v, \sin v)$. The last displayed inequalities now imply

$$|\cos u - \cos v| \le \overline{AB} \qquad \text{and} \qquad |\sin u - \sin v| \le \overline{AB}.$$

Recall that the points A and B are on the unit circle. Any two points on the unit circle determine two arcs. Denote by $\widehat{\mathrm{AB}}$ the length of the shorter circular arc determined by A and B. Since the shortest path between two points is a straight line we have that $\overline{AB} < \widehat{\mathrm{AB}}$. How is the arc length $\widehat{\mathrm{AB}}$ related to the numbers u and v? First, if $|u - v| \le \pi$, then $\widehat{\mathrm{AB}} = |u - v|$. Second, if $|u - v| > \pi$, then $\widehat{\mathrm{AB}} \le \pi < |u - v|$. Hence in each case $\widehat{\mathrm{AB}} \le |u - v|$. Thus we have established inequalities

$$|\cos u - \cos v| \le \overline{AB} \le \widehat{\mathrm{AB}} \le |u - v|,$$
$$|\sin u - \sin v| \le \overline{AB} \le \widehat{\mathrm{AB}} \le |u - v|,$$

for arbitrary real numbers u and v. These inequalities can be used to solve Exercises 6.10 and 6.11. THE END OF THE HINT.

EXAMPLE 6.12. Let $f(x) = \ln x$ for all $x \in (0, +\infty)$. Prove that f is continuous on its domain.

SOLUTION. First we recall the inequality

$$1 - \frac{1}{v} \le \ln v \le v - 1 \qquad \text{valid for all} \qquad v > 0, \tag{6.1}$$

which we proved using the integral definition of ln.

An inequality for $|\ln v|$ will be useful in the proof of the continuity below. Such an inequality can be obtained from the inequality in (6.2) by considering two cases:

$$\begin{aligned} |\ln v| &\le \left\{ \begin{array}{ll} v - 1 & \text{if} \;\; 1 \le v \\ -\left(1 - \frac{1}{v}\right) & \text{if} \;\; 0 < v < 1 \end{array} \right\} \\ &= \left\{ \begin{array}{ll} v - 1 & \text{if} \;\; 1 \le v \\ -\frac{v-1}{v} & \text{if} \;\; 0 < v < 1 \end{array} \right\} \\ &= \left\{ \begin{array}{ll} |v - 1| & \text{if} \;\; 1 \le v \\ \frac{|v-1|}{v} & \text{if} \;\; 0 < v < 1 \end{array} \right\}. \end{aligned}$$

Next we will restrict v to the interval $(1/2, 3/2)$. That is we assume $v \in (1/2, 3/2)$. Then we have that $|v - 1|/v \le 2|v - 1|$. Since always $|v - 1| \le 2|v - 1|$, we have that

$$|\ln v| \le 2|v - 1| \qquad \text{is valid for all} \qquad v \in (1/2, 3/2). \tag{6.2}$$

Let $a > 0$ be arbitrary. Let $x \in (a/2, 3a/2)$. Then $x/a \in (1/2, 3/2)$ and we can simplify the expression $|\ln x - \ln a|$ which appears in the definition of continuity. In the next sequence of inequalities we first use a property of logarithm, then the inequality in (6.2) and simple algebra to get:

$$\begin{aligned} |\ln x - \ln a| &= \left|\ln \frac{x}{a}\right| \\ &\le 2\left|\frac{x}{a} - 1\right| \\ &= 2\left|\frac{x-a}{a}\right| \\ &= 2\frac{|x-a|}{a} \\ &= \frac{2}{a}\,|x-a|. \end{aligned}$$

Thus, we proved that

$$|\ln x - \ln a| \le \frac{2}{a}\,|x-a| \qquad \text{is valid for all} \qquad x \in (a/2, 3a/2). \tag{6.3}$$

To finish the proof of continuity let $\epsilon > 0$ be arbitrary and set

$$\delta(\epsilon) = \min\left\{\frac{a\epsilon}{2}, \frac{a}{2}\right\}.$$

Clearly $\delta(\epsilon) > 0$. Next we will prove the implication

$$|x-a| < \min\left\{\frac{a\epsilon}{2}, \frac{a}{2}\right\} \qquad \Rightarrow \qquad |\ln x - \ln a| < \epsilon.$$

Assume $|x-a| < \min\left\{\frac{a\epsilon}{2}, \frac{a}{2}\right\}$. Then $|x-a| < \frac{a\epsilon}{2}$ and $|x-a| < \frac{a}{2}$. Since $|x-a| < \frac{a}{2}$, we have $x \in (a/2, 3a/2)$ and therefore, by (6.3), we have

$$|\ln x - \ln a| \le \frac{2}{a}\,|x-a|.$$

Since $|x-a| < \frac{a\epsilon}{2}$ we have

$$\frac{2}{a}\,|x-a| < \epsilon.$$

The last two displayed inequalities yield

$$|\ln x - \ln a| < \epsilon.$$

This completes the proof of the continuity of the logarithm function ln. □

EXAMPLE 6.13. Let $f(x) = e^x$ for all $x \in \mathbb{R}$. Prove that f is continuous at an arbitrary real number a.

SOLUTION. We first substitute $v = \exp u = e^u$ in (6.1) to get

$$1 - \frac{1}{e^u} \le \ln e^u \le e^u - 1 \qquad \text{is valid for all} \qquad u \in \mathbb{R}.$$

Simplifying we get

$$1 - \frac{1}{e^u} \le u \le e^u - 1.$$

We need a squeeze for e^u. Above we already have one side of the squeeze. That is $u+1 \le e^u$. To get the other side we transform

$$1 - \frac{1}{e^u} \le u$$

to

$$1 - u \leq \frac{1}{e^u}.$$

To get a useful inequality we need to take the reciprocals in the last inequality. For that we need $1 - u > 0$. That is we need to assume that $u < 1$. Assuming that $u < 1$ we have

$$e^u \leq \frac{1}{1-u}.$$

Together with $u + 1 \leq e^u$, we proved that

$$u + 1 \leq e^u \leq \frac{1}{1-u} \qquad \text{is valid for all} \qquad u < 1. \tag{6.4}$$

An inequality for $|e^u - 1|$ will be useful in the proof of the continuity below. The inequality in (6.4) yields that

$$u \leq e^u - 1 \leq \frac{u}{1-u} \qquad \text{is valid for all} \qquad u < 1.$$

To get an inequality for $|e^u - 1|$ we consider two cases:

$$\begin{aligned} |e^u - 1| &\leq \left\{ \begin{array}{ll} \frac{u}{1-u} & \text{if} \quad 0 \leq u < 1 \\ -u & \text{if} \quad u < 0 \end{array} \right\} \\ &= \left\{ \begin{array}{ll} \frac{|u|}{1-u} & \text{if} \quad 0 \leq u < 1 \\ |u| & \text{if} \quad u < 0 \end{array} \right\} \end{aligned}$$

Next we will restrict u to the interval $(-1/2, 1/2)$. That is we assume $u \in (-1/2, 1/2)$. Then we have that $|u|/(1-u) \leq 2|u|$. Since always $|u| \leq 2|u|$, we have that

$$|e^u - 1| \leq 2|u| \qquad \text{is valid for all} \qquad u \in (-1/2, 1/2). \tag{6.5}$$

Let $a > 0$ be arbitrary. Let $x \in (a - 1/2, a + 1/2)$. Then $x - a \in (-1/2, 1/2)$ and we can simplify the expression $|e^x - e^a|$ which appears in the definition of continuity. For that we use a property of the exponential function and (6.5) to get:

$$|e^x - e^a| = e^a |e^{(x-a)} - 1| \leq 2e^a |x - a|.$$

Thus, we proved that

$$|e^x - e^a| \leq 2e^a |x - a| \qquad \text{is valid for all} \qquad x \in (a - 1/2, a + 1/2). \tag{6.6}$$

To finish the proof of the continuity let $\epsilon > 0$ be arbitrary and set

$$\delta(\epsilon) = \min\left\{ \frac{\epsilon}{2e^a}, \frac{1}{2} \right\}.$$

Clearly $\delta(\epsilon) > 0$.

Next we will prove the implication

$$|x - a| < \min\left\{ \frac{\epsilon}{2e^a}, \frac{1}{2} \right\} \qquad \Rightarrow \qquad |e^x - e^a| < \epsilon.$$

Assume $|x - a| < \min\left\{ \frac{\epsilon}{2e^a}, \frac{1}{2} \right\}$. Then $|x - a| < \frac{\epsilon}{2e^a}$ and $|x - a| < \frac{1}{2}$. Since $|x - a| < \frac{1}{2}$, we have $x \in (a - 1/2, a + 1/2)$ and therefore, by (6.6), we have

$$|e^x - e^a| \leq 2e^a |x - a|.$$

Since $|x - a| < \frac{\epsilon}{2e^a}$ we have

$$2e^a |x - a| < \epsilon.$$

The last two displayed inequalities yield

$$|e^x - e^a| < \epsilon.$$

This completes the proof of the continuity of of the exponential function exp. □

6.2. General theorems about continuous functions. The following theorem follows from Theorem 5.7.

THEOREM 6.14 (Algebra of Continuous Functions). *Let f and g be functions and let a be a real number. Assume that f and g are continuous at the point a.*

(a) *If $h = f + g$, then h is continuous at a.*
(b) *If $h = fg$, then h is continuous at a.*
(c) *If $h = \dfrac{f}{g}$ and $g(a) \neq 0$, then h is continuous at a.*

EXAMPLE 6.15. Let $f(x) = \tan x$ for all $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$. Prove that f is continuous at an arbitrary real number a such that $-\dfrac{\pi}{2} < a < \dfrac{\pi}{2}$.

SOLUTION. Use the algebra of continuous functions. □

The following theorem states that a composition of continuous functions is continuous.

THEOREM 6.16. *Let f and g be functions and let a be a real number. Assume that g is continuous at a and that f is continuous at $g(a)$. If $h = f \circ g$, then h is continuous at a.*

PROOF. Assume that the function g is continuous at a. That is assume

(I-g) There exists a $\delta_{0,g} > 0$ such that $g(x)$ is defined for all $x \in (a - \delta_{0,g}, a + \delta_{0,g})$.
(II-g) For each $\epsilon > 0$ there exists $\delta_g(\epsilon)$ such that $0 < \delta_g(\epsilon) \leq \delta_{0,g}$ and such that

$$|x - a| < \delta_g(\epsilon) \quad \Rightarrow \quad |g(x) - g(a)| < \epsilon.$$

Also assume that the function f is continuous at $g(a)$. That is assume

(I-f) There exists a $\delta_{0,f} > 0$ such that $f(x)$ is defined for all $x \in \big(g(a) - \delta_{0,f}, g(a) + \delta_{0,f}\big)$.
(II-f) For each $\epsilon > 0$ there exists $\delta_g(\epsilon)$ such that $0 < \delta_f(\epsilon) \leq \delta_{0,f}$ and such that

$$|u - g(a)| < \delta_f(\epsilon) \quad \Rightarrow \quad |f(u) - f(g(a))| < \epsilon.$$

Let $h = f \circ g$, that is $h(x) = f(g(x))$. I have to prove that h has the following properties: (These items are red.)

(I-h) There exists a $\delta_{0,h} > 0$ such that $h(x)$ is defined for all $x \in \big(a - \delta_{0,h}, a + \delta_{0,h}\big)$.
(II-h) For each $\epsilon > 0$ there exists $\delta_h(\epsilon)$ such that $0 < \delta_h(\epsilon) \leq \delta_{0,h}$ and such that

$$|x - a| < \delta_h(\epsilon) \quad \Rightarrow \quad |h(x) - h(a)| < \epsilon.$$

Where is h guaranteed to be defined? I must make sure that x is such that $|g(x) - g(a)| < \delta_{0,f}$. We can achieve this by using (II-g)!

Put $\delta_{0,h} := \delta_g\big(\delta_{0,f}\big)$. Now assume that $|x - a| < \delta_{0,h}$. By (II-g) it follows that $|g(x) - g(a)| < \delta_{0,f}$. Therefore $g(x) \in \big(g(a) - \delta_{0,f}, g(a) + \delta_{0,f}\big)$. Hence, by (I-f), $f(g(x))$ is defined. Thus we proved that $f(g(x))$ is defined whenever $|x - a| < \delta_{0,h}$.

Let $\epsilon > 0$ be given. Put

$$\delta_h(\epsilon) := \min\big\{\delta_g\big(\delta_f(\epsilon)\big), \delta_g\big(\delta_{0,f}\big)\big\}.$$

Now we prove the red implication in (II-h).

Assume $|x-a| < \delta_h(\epsilon)$. Then $|x-a| < \delta_g\big(\delta_f(\epsilon)\big)$. By the green implication in (II-g), we conclude that

$$|x-a| < \delta_g\big(\delta_f(\epsilon)\big) \quad \Rightarrow \quad |g(x)-g(a)| < \delta_f(\epsilon).$$

Using the green implication in (II-f), we conclude that

$$|g(x)-g(a)| < \delta_f(\epsilon) \quad \Rightarrow \quad |f(g(x))-f(g(a))| < \epsilon.$$

Thus we proved that the assumption $|x-a| < \delta_h(\epsilon)$ implies that

$$|h(x)-h(a)| = |f(g(x))-f(g(a))| < \epsilon.$$

This completes the proof. □

CHAPTER 2

Infinite Series

1. Sequences of real numbers

1.1. Definitions and examples.

DEFINITION 1.1. A sequence of real numbers is a real function whose domain is the set $\mathbb{N}$ of natural numbers.

Let $s : \mathbb{N} \to \mathbb{R}$ be a sequence. Then the values of s are $s(1), s(2), s(3), \dots, s(n), \dots$. It is customary to write s_n instead of $s(n)$ in this case. Sometimes a sequence will be specified by listing its first few terms

$$s_1,\ s_2,\ s_3,\ s_4,\ \dots\ ,$$

and sometimes by listing of all its terms $\{s_n\}_{n\in\mathbb{N}}$ or $\{s_n\}_{n=1}^{+\infty}$. One way of specifying a sequence is to give a formula, or recursion formula for its $n-$th term s_n. Notice that in this notation s is the "name" of the sequence and n is the variable.

Some examples of sequences follow.

EXAMPLE 1.2. (a) $1,\ 0,\ -1,\ 0,\ 1,\ 0,\ -1,\ \dots$;

(b) $1,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4,\ 4,\ 4,\ 5,\ 5,\ 5,\ 5,\ 5,\ 6,\ 6,\ 6,\ 6,\ 6,\ 6,\ 7,\ 7,\ \dots$;

(c) $1,\ 1,\ 1,\ 1,\ 1,\ \dots$; (the constant sequence)

(d) $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \dots$; (What is the range of this sequence?)

Recursively defined sequences

EXAMPLE 1.3. (a) $x_1 = 1, \quad x_{n+1} = 1 + \frac{x_n}{4}, \ n = 1, 2, 3, \dots$;

(b) $x_1 = 2, \quad x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}, \ n = 1, 2, 3, \dots$;

(c) $a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2 + a_n}, \quad n = 1, 2, 3, \dots$;

(d) $s_1 = 1, \quad s_{n+1} = \sqrt{1 + s_n}, \quad n = 1, 2, 3, \dots$;

(e) $x_1 = \frac{9}{10}, \quad x_{n+1} = \frac{9 + x_n}{10}, \quad n = 1, 2, 3, \dots$.

(f) $b_1 = \frac{1}{2}, \quad b_{n+1} = \frac{1}{2\sqrt{1 - b_n^2}}, \quad n = 1, 2, 3, \dots$

(g) $f_1 = 1, \ f_{n+1} = (n+1)\, f_n, \quad n = 1, 2, 3, \dots$.

Some important examples of sequences are listed below.

$$b_n = c, \quad c \in \mathbb{R}. \ \ n \in \mathbb{N}, \tag{1.1}$$

$$p_n = a^n, \quad a \in \mathbb{R}, \quad n \in \mathbb{N}, \tag{1.2}$$

$$x_n = \left(1 + \frac{1}{n}\right)^n, \quad n \in \mathbb{N}, \tag{1.3}$$

$$y_n = \left(1 + \frac{1}{n}\right)^{(n+1)}, \quad n \in \mathbb{N}, \tag{1.4}$$

$$z_n = \left(1 + \frac{a}{n}\right)^n, \quad a \in \mathbb{R}, \quad n \in \mathbb{N}, \tag{1.5}$$

$$f_1 = 1, \quad f_{n+1} = f_n \cdot (n+1), \quad n \in \mathbb{N}. \tag{1.6}$$

(The standard notation for the terms of the sequence $\{f_n\}_{n=1}^{+\infty}$ is $f_n = n!,\ n \in \mathbb{N}$.)

$$q_n = \frac{a^n}{n!}, \quad a \in \mathbb{R}, \quad n \in \mathbb{N}, \tag{1.7}$$

$$t_1 = 2,\ t_{n+1} = t_n + \frac{1}{(n+1)!}, \quad n \in \mathbb{N}, \tag{1.8}$$

$$v_1 = 1 + a,\ v_{n+1} = v_n + \frac{a^n}{(n+1)!}, \quad a \in \mathbb{R}, \quad n \in \mathbb{N}. \tag{1.9}$$

Let $\{a_n\}_{n=1}^{+\infty}$ be an arbitrary sequence. An important sequence associated with $\{a_n\}_{n=1}^{+\infty}$ is the following sequence

$$S_1 = a_1,\ S_{n+1} = S_n + a_{n+1}, \quad n \in \mathbb{N}. \tag{1.10}$$

1.2. Convergent sequences.

DEFINITION 1.4. A sequence $\{s_n\}_{n=1}^{+\infty}$ of real numbers *converges* to the real number L if for each $\epsilon > 0$ there exists a number $N(\epsilon)$ such that

$$n \in \mathbb{N}, \quad n > N(\epsilon) \qquad \Rightarrow \qquad |s_n - L| < \epsilon.$$

If $\{s_n\}_{n=1}^{+\infty}$ converges to L we will write

$$\lim_{n\to+\infty} s_n = L \qquad \text{or} \qquad s_n \to L \quad (n \to +\infty).$$

The number L is called the *limit* of the sequence $\{s_n\}_{n=1}^{+\infty}$. A sequence that does not converge to a real number is said to *diverge*.

EXAMPLE 1.5. Let r be a real number such that $|r| < 1$. Prove that $\lim_{n\to+\infty} r^n = 0$.

SOLUTION. First note that if $r = 0$, then $r^n = 0$ for all $n \in \mathbb{N}$, so the given sequence is a constant sequence. Therefore it converges. Let $\epsilon > 0$. We need to solve $|r^n - 0| < \epsilon$ for n. First simplify $|r^n - 0| = |r^n| = |r|^n$. Now solve $|r|^n < \epsilon$ by taking ln of both sides of the inequality (note that ln is an increasing function)

$$\ln |r|^n = n \ln |r| < \ln \epsilon.$$

Since $|r| < 1$, we conclude that $\ln |r| < 0$. Therefore the solution is $\quad n > \dfrac{\ln \epsilon}{\ln |r|}$. Thus, with $N(\epsilon) = \dfrac{\ln \epsilon}{\ln |r|}$, the implication

$$n \in \mathbb{N}, \quad n > N(\epsilon) \quad \Rightarrow \quad |r^n - 0| < \epsilon$$

is valid. □

EXAMPLE 1.6. Prove that $\displaystyle\lim_{n\to+\infty} \frac{n^2 - n - 1}{2n^2 - 1} = \frac{1}{2}$.

SOLUTION. Let $\epsilon > 0$ be given. We need to solve $\left|\frac{n^2-n-1}{2n^2-1} - \frac{1}{2}\right| < \epsilon$ for n. First simplify:

$$\left|\frac{n^2-n-1}{2n^2-1} - \frac{1}{2}\right| = \left|\frac{2}{2}\,\frac{n^2-n-1}{2n^2-1} - \frac{1}{2}\,\frac{2n^2-1}{2n^2-1}\right| = \left|\frac{-2n-1}{2\,(2n^2-1)}\right| = \frac{2n+1}{4n^2-2}$$

Now invent the BIN:

$$\frac{2n+1}{4n^2-2} \leq \frac{2n+n}{4n^2-2n^2} = \frac{3n}{2n^2} = \frac{3}{2n}.$$

Therefore the BIN is:

$$\left|\frac{n^2-n-1}{2n^2-1} - \frac{1}{2}\right| \leq \frac{3}{2n} \quad \text{valid for} \quad n \in \mathbb{N}.$$

Solving for n is now easy:

$$\frac{3}{2n} < \epsilon. \quad \text{The solution is} \quad n > \frac{3}{2\epsilon}.$$

Thus, with $N(\epsilon) = \frac{3}{2\epsilon}$, the implication

$$n > N(\epsilon) \quad \Rightarrow \quad \left|\frac{n^2-n-1}{2n^2-1} - \frac{1}{2}\right| < \epsilon$$

is valid. Using the BIN, this implication should be easy to prove. □

This procedure is very similar to the procedure for proving limits as x approaches infinity. In fact the following two theorems are true.

THEOREM 1.7. *Let $x \mapsto f(x)$ be a function which is defined for every $x \geq 1$. Define the sequence $a : \mathbb{N} \to \mathbb{R}$ by*

$$a_n = f(n) \quad \textit{for every} \quad n \in \mathbb{N}.$$

If $\lim_{x\to+\infty} f(x) = L$, then $\lim_{n\to+\infty} a_n = L$.

THEOREM 1.8. *Let $x \mapsto f(x)$ be a function which is defined for every $x \in (0, 1]$. Define the sequence $a : \mathbb{N} \to \mathbb{R}$ by*

$$a_n = f(1/n) \quad \textit{for every} \quad n \in \mathbb{N}.$$

If $\lim_{x\downarrow 0} f(x) = L$, then $\lim_{n\to+\infty} a_n = L$.

The above two theorems are useful for proving limits of sequences which are defined by a formula. For example you can prove the following limits by using these two theorems and what we proved in previous sections.

EXERCISE 1.9. Find the following limits. Provide proofs.

(a) $\lim_{n\to+\infty} \sin\left(\frac{1}{n}\right)$ (b) $\lim_{n\to+\infty} n \sin\left(\frac{1}{n}\right)$ (c) $\lim_{n\to+\infty} \ln\left(1 + \frac{1}{n}\right)$

(d) $\lim_{n\to+\infty} n \ln\left(1 + \frac{1}{n}\right)$ (e) $\lim_{n\to+\infty} \cos\left(\frac{1}{n}\right)$ (f) $\lim_{n\to+\infty} \frac{1}{n} \cos\left(\frac{1}{n}\right)$

The Algebra of Limits Theorem holds for sequences.

THEOREM 1.10. *Let $\{a_n\}_{n=1}^{+\infty}$, $\{b_n\}_{n=1}^{+\infty}$ and $\{c_n\}_{n=1}^{+\infty}$, be given sequences. Let K and L be real numbers. Assume that*

(1) $\lim_{x\to+\infty} a_n = K$,
(2) $\lim_{x\to+\infty} b_n = L$.

Then the following statements hold.

(A) *If* $c_n = a_n + b_n$, $n \in \mathbb{N}$, *then* $\lim_{x\to+\infty} c_n = K + L$.
(B) *If* $c_n = a_n b_n$, $n \in \mathbb{N}$, *then* $\lim_{x\to+\infty} c_n = KL$.
(C) *If* $L \neq 0$ *and* $c_n = \dfrac{a_n}{b_n}$, $n \in \mathbb{N}$, *then* $\lim_{x\to+\infty} c_n = \dfrac{K}{L}$.

THEOREM 1.11. *Let* $\{a_n\}_{n=1}^{+\infty}$ *and* $\{b_n\}_{n=1}^{+\infty}$ *be given sequences. Let* K *and* L *be real numbers. Assume that*

(1) $\lim_{x\to+\infty} a_n = K$.
(2) $\lim_{x\to+\infty} b_n = L$.
(3) *There exists a natural number* n_0 *such that*

$$a_n \leq b_n \quad \text{for all} \quad n \geq n_0.$$

Then $K \leq L$.

THEOREM 1.12. *Let* $\{a_n\}_{n=1}^{+\infty}$, $\{b_n\}_{n=1}^{+\infty}$ *and* $\{s_n\}_{n=1}^{+\infty}$ *be given sequences. Assume the following*

(1) *The sequence* $\{a_n\}_{n=1}^{+\infty}$ *converges to the limit* L.
(2) *The sequence* $\{b_n\}_{n=1}^{+\infty}$ *converges to the limit* L.
(3) *There exists a natural number* n_0 *such that*

$$a_n \leq s_n \leq b_n \quad \text{for all} \quad n > n_0.$$

Then the sequence $\{s_n\}_{n=1}^{+\infty}$ *converges to the limit* L.

Prove this theorem.

1.3. The Monotone Convergence Theorem. Many limits of sequences cannot be found using theorems from the previous section. For example, the recursively defined sequences (a), (b), (c), (d) and (e) in Example 1.3 converge but it cannot be proved using the methods that we presented so far.

DEFINITION 1.13. Let $\{s_n\}_{n=1}^{+\infty}$ be a sequence of real numbers.

(1) If a real number M satisfies

$$s_n \leq M \quad \text{for all} \quad n \in \mathbb{N}$$

then M is called an *upper bound* of $\{s_n\}_{n=1}^{+\infty}$ and the sequence $\{s_n\}_{n=1}^{+\infty}$ is said to be *bounded above*.

(2) If a real number m satisfies

$$m \leq s_n \quad \text{for all} \quad n \in \mathbb{N},$$

then m is called a *lower bound* of $\{s_n\}_{n=1}^{+\infty}$ and the sequence $\{s_n\}_{n=1}^{+\infty}$ is said to be *bounded below*.

(3) The sequence $\{s_n\}_{n=1}^{+\infty}$ is said to be *bounded* if it is bounded above and bounded below.

THEOREM 1.14. *If a sequence converges, then it is bounded.*

PROOF. Assume that a sequence $\{a_n\}_{n=1}^{+\infty}$ converges to L. By Definition 1.4 this means that for each $\epsilon > 0$ there exists a number $N(\epsilon)$ such that

$$n \in \mathbb{N}, \quad n > N(\epsilon) \qquad \Rightarrow \qquad |a_n - L| < \epsilon.$$

In particular for $\epsilon = 1 > 0$ there exists a number $N(1)$ such that

$$n \in \mathbb{N}, \quad n > N(1) \qquad \Rightarrow \qquad |a_n - L| < 1.$$

Let n_0 be the largest natural number which is $\leq \ N(1)$. Then $n_0 + 1, n_0 + 2, \ldots$ are all $> \ N(1)$. Therefore

$$|a_n - L| < 1 \qquad \text{for all} \qquad n > n_0.$$

This means that

$$L - 1 < a_n < L + 1 \qquad \text{for all} \qquad n > n_0.$$

The numbers $L-1$ and $L+1$ are not lower and upper bounds for the sequence since we do not know how they relate to the first n_0 terms of the sequence. Put

$$\begin{aligned} m &= \min\{a_1, a_2, \ldots, a_{n_0}, L - 1\} \\ M &= \max\{a_1, a_2, \ldots, a_{n_0}, L + 1\}. \end{aligned}$$

Clearly

$$\begin{aligned} m \leq a_n &\qquad \text{for all} \qquad n = 1, 2, \ldots, n_0 \\ m \leq L - 1 < a_n &\qquad \text{for all} \qquad n > n_0. \end{aligned}$$

Thus m is a lower bound for the sequence $\{a_n\}_{n=1}^{+\infty}$.

Clearly

$$\begin{aligned} a_n \leq & M \qquad \text{for all} \qquad n = 1, 2, \ldots, n_0 \\ a_n < L + 1 \leq & M \qquad \text{for all} \qquad n > n_0. \end{aligned}$$

Thus M is an upper bound for the sequence $\{a_n\}_{n=1}^{+\infty}$. □

Is the converse of Theorem 1.14 true? The converse is: If a sequence is bounded, then it converges. Clearly a counterexample to the last implication is the sequence $(-1)^n, n \in \mathbb{N}$. This sequence is bounded but it is not convergent.

The next question is whether boundedness and an additional property of a sequence can guarantee convergence. It turns out that such an property is monotonicity defined in the following definition.

DEFINITION 1.15. A sequence $\{s_n\}_{n=1}^{+\infty}$ of real numbers is said to be

non-decreasing if $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$,

strictly increasing if $s_n < s_{n+1}$ for all $n \in \mathbb{N}$,

non-increasing if $s_n \geq s_{n+1}$ for all $n \in \mathbb{N}$.

strictly decreasing if $s_n > s_{n+1}$ for all $n \in \mathbb{N}$.

A sequence with either of these four properties is said to be *monotonic*.

The following two theorems give powerful tools for establishing convergence of a sequence.

THEOREM 1.16. *If $\{s_n\}_{n=1}^{+\infty}$ is non-decreasing and bounded above, then $\{s_n\}_{n=1}^{+\infty}$ converges.*

THEOREM 1.17. *If* $\{s_n\}_{n=1}^{+\infty}$ *is non-increasing and bounded below, then* $\{s_n\}_{n=1}^{+\infty}$ *converges.*

To prove these theorems we have to resort to the most important property of the set of real numbers: the Completeness Axiom.

THE COMPLETENESS AXIOM. If A and B are nonempty subsets of $\mathbb{R}$ such that for every $a \in A$ and for every $b \in B$ we have $a \le b$, then there exists $c \in \mathbb{R}$ such that $a \le c \le b$ for all $a \in A$ and all $b \in B$.

PROOF OF THEOREM 1.16. Assume that $\{s_n\}_{n=1}^{+\infty}$ is a non-decreasing sequence and that it is bounded above. Since $\{s_n\}_{n=1}^{+\infty}$ is non-decreasing we know that

$$s_1 \le s_2 \le s_3 \le \cdots \le s_{n-1} \le s_n \le s_{n+1} \le \cdots . \tag{1.11}$$

Let A be the range of the sequence $\{s_n\}_{n=1}^{+\infty}$. That is $A = \big\{s_n : n \in \mathbb{N}\big\}$. Clearly $A \neq \emptyset$. Let B be the set of all upper bounds of the sequence $\{s_n\}_{n=1}^{+\infty}$. Since the sequence $\{s_n\}_{n=1}^{+\infty}$ is bounded above, the set B is not empty. Let $b \in B$ be arbitrary. Then b is an upper bound for $\{s_n\}_{n=1}^{+\infty}$. Therefore

$$s_n \le b \qquad \text{for all} \qquad n \in \mathbb{N}.$$

By the definition of A this means

$$a \le b \qquad \text{for all} \qquad a \in A.$$

Since $b \in B$ was arbitrary we have

$$a \le b \quad \text{for all} \quad a \in A \quad \text{and for all} \quad b \in B.$$

By the Completeness Axiom there exists $c \in \mathbb{R}$ such that

$$s_n \le c \le b \qquad \text{for all} \quad n \in \mathbb{N} \quad \text{and for all} \quad b \in B. \tag{1.12}$$

Thus c is an upper bound for $\{s_n\}_{n=1}^{+\infty}$ and also $c \le b$ for all upper bounds b of the sequence $\{s_n\}_{n=1}^{+\infty}$. Therefore, for an arbitrary $\epsilon > 0$ the number $c - \epsilon$ (which is $< c$) is not an upper bound of the sequence $\{s_n\}_{n=1}^{+\infty}$. Consequently, there exists a natural number $N(\epsilon)$ such that

$$c - \epsilon < s_{N(\epsilon)}. \tag{1.13}$$

Let $n \in \mathbb{N}$ be any natural number which is $> N(\epsilon)$. Then the inequalities (1.11) imply that

$$s_{N(\epsilon)} \le s_n. \tag{1.14}$$

By (1.12) the number c is an upper bound of $\{s_n\}_{n=1}^{+\infty}$. Hence we have

$$s_n \le c \qquad \text{for all} \qquad n \in \mathbb{N}. \tag{1.15}$$

Putting together the inequalities (1.13), (1.14) and (1.15) we conclude that

$$c - \epsilon < s_n \le c \quad \text{for all} \quad n \in \mathbb{N} \ \text{ such that } \ n > N(\epsilon). \tag{1.16}$$

The relationship (1.16) shows that for $n \in \mathbb{N}$ such that $n > N(\epsilon)$ the distance between numbers s_n and c is $< \epsilon$. In other words

$$n \in \mathbb{N}, \quad n > N(\epsilon) \quad \text{implies} \quad |s_n - c| < \epsilon.$$

This is exactly the implication in Definition 1.4. Thus, we proved that

$$\lim_{n \to +\infty} s_n = c. \qquad \square$$

EXAMPLE 1.18. Prove that the sequence in Example 1.3 (b) converges. That is, prove that the recursively defined sequence $x_1 = 2, \quad x_{n+1} = \dfrac{x_n}{2} + \dfrac{1}{x_n}$, $n \in \mathbb{N}$, converges.

SOLUTION. It is useful to calculate the first few terms of this sequence:

$$x_1 = 2,\ x_2 = \frac{3}{2},\ x_3 = \frac{17}{12},\ x_4 = \frac{577}{408},\ x_5 = \frac{665857}{470832},\ x_6 = \frac{886731088897}{627013566048}.$$

Notice that the formula $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ gives a positive output x_{n+1} whenever the input x_n is positive. Since $x_1 > 0$ this guaranties that $x_2 > 0$. In turn, the fact that $x_2 > 0$ guaranties that $x_3 > 0$, and so on. This reasoning justifies that $x_n > 0$ for all $n \in \mathbb{N}$. This proves that the sequence $\{x_n\}$ is bounded below by 0.

Next we will prove that $\bigl(x_n\bigr)^2 \geq 2$ for all $n \in \mathbb{N}$. We consider two cases $n = 1$ and $n > 1$. If $n = 1$, then $\bigl(x_1\bigr)^2 = 2^2 = 4 \geq 2$. Now assume that $n > 1$. Then $n - 1 \in \mathbb{N}$ and $x_n = \frac{x_{n-1}}{2} + \frac{1}{x_{n-1}}$. Therefore

$$\begin{aligned}\bigl(x_n\bigr)^2 &= \Bigl(\frac{x_{n-1}}{2} + \frac{1}{x_{n-1}}\Bigr)^2\\ &= \frac{(x_{n-1})^2}{4} + 1 + \frac{1}{(x_{n-1})^2}\\ &= 2 + \frac{(x_{n-1})^2}{4} - 1 + \frac{1}{(x_{n-1})^2}\\ &= 2 + \Bigl(\frac{x_{n-1}}{2} - \frac{1}{x_{n-1}}\Bigr)^2\\ &\geq 2.\end{aligned}$$

Thus $\bigl(x_n\bigr)^2 \geq 2$ for all $n \in \mathbb{N}$.

Since $x_n > 0$, $\bigl(x_n\bigr)^2 \geq 2$ implies $x_n \geq \frac{2}{x_n}$. Further, dividing by 2 we get $\frac{x_n}{2} \geq \frac{1}{x_n}$. Adding $\frac{x_n}{2}$ to the both sides of the last inequality we obtain $x_n \geq \frac{x_n}{2} + \frac{1}{x_n}$. Thus $x_n \geq x_{n+1}$. Here we have proved that $\bigl(x_n\bigr)^2 \geq 2$ implies $x_n \geq x_{n+1}$. Since $\bigl(x_n\bigr)^2 \geq 2$ is true for all $n \in \mathbb{N}$, we have proved that $x_n \geq x_{n+1}$ is true for all $n \in \mathbb{N}$.

To summarize, we have proved that $x_n > 0$ for all $n \in \mathbb{N}$ and $x_n \geq x_{n+1}$ is true for all $n \in \mathbb{N}$. That $\{x_n\}$ is bounded below and non-increasing. By the Monotone Convergence Theorem this sequence converges. Denote the limit of $\{x_n\}$ by L.

Next we use the algebra of limits to calculate L. Since $\bigl(x_n\bigr)^2 \geq 2$ for all $n \in \mathbb{N}$, by Theorems 1.10 and 1.11 we have $L^2 \geq 2$. Since $x_n > 0$ for all $n \in \mathbb{N}$, by Theorem 1.11 we have $L \geq 0$. Since $L^2 \geq 0$ and $L \geq 0$ we have $L > 0$. It is not difficult to prove that $\lim_{n\to\infty} x_{n+1} = L$. This fact, Theorem 1.10 and the identity $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ imply $L = \frac{L}{2} + \frac{1}{L}$. Hence $L^2 = 2$. That is $L = \sqrt{2}$.

This example is in fact a proof that there exists a positive real number a such that $a^2 = 2$. ☐

EXAMPLE 1.19. Prove that the sequence $T_n = \displaystyle\sum_{k=0}^{n} \frac{1}{n!}$, $n \in \mathbb{N}$, converges.

SOLUTION. Let $n \in \mathbb{N}$. We first prove that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}. \tag{1.17}$$

Set $A = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}$. Then $\frac{1}{2}A = \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n+1}}$. Therefore $\frac{1}{2}A = A - \frac{1}{2}A = \frac{1}{2} - \frac{1}{2^{n+1}}$. Multiplying by 2 both sides of $\frac{1}{2}A = \frac{1}{2} - \frac{1}{2^{n+1}}$ we get $A = 1 - \frac{1}{2^n}$ which is (1.17).

For $n \in \mathbb{N}$ we have $n! \geq 2^{n-1}$. Therefore for $n > 1$ we have

$$\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} = 1 - \frac{1}{2^{n-1}}.$$

Consequently, for $n > 1$,

$$\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} < 1.$$

Therefore for all $n \in \mathbb{N}$

$$T_n = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} < 3.$$

This proves that the sequence $\{T_n\}$ is bounded above. Since for every $n \in \mathbb{N}$ we have $T_{n+1} - T_n = \dfrac{1}{(n+1)!} > 0$, the sequence $\{T_n\}$ is increasing. By the Monotone Convergence Theorem $\{T_n\}$ converges.

The limit of the sequence $\{T_n\}$ is the famous number e. □

Example 1.20. Prove that the sequence

$$t_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln n, \qquad n \in \mathbb{N},$$

converges.

Solution. Let $n \in \mathbb{N}$. By the definition

$$\ln n = \int_1^n \frac{1}{x}\, dx.$$

Since

$$\frac{1}{x} \leq \frac{1}{k} \qquad \text{whenever} \qquad k \leq x \leq k+1,$$

for $n > 1$ we have

$$\ln n = \int_1^2 \frac{1}{x}\, dx + \int_2^3 \frac{1}{x}\, dx + \cdots + \int_{n-1}^n \frac{1}{x}\, dx < 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-1}.$$

Therefore,

$$t_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-1} + \frac{1}{n} - \ln n > \frac{1}{n} > 0$$

for all $n \in \mathbb{N}, n > 1$. Since $t_1 = 1 > 0$, this proves that the sequence $\{t_n\}$ is bounded below by 0.

Next we prove that $\{t_n\}$ is decreasing. For arbitrary $n \in \mathbb{N}$ we have

$$\begin{aligned} t_n - t_{n+1} &= \big(\ln(n+1) - \ln n\big) - \frac{1}{n+1} \\ &= \int_n^{n+1} \frac{1}{x}\, dx - \frac{1}{n+1} \\ &= \int_n^{n+1} \left(\frac{1}{x} - \frac{1}{n+1}\right) dx \\ &> 0. \end{aligned}$$

Hence $t_n > t_{n+1}$ for all $n \in \mathbb{N}$.

Since $\{t_n\}$ is bounded below and decreasing it converges by the Monotone Convergence Theorem.

The limit of the sequence $\{t_n\}$ is called *Euler's constant*. It is denoted by γ. Its approximate value to 50 decimal places is

$$\gamma \approx 0.57721566490153286060651209008240243104215933593992.$$

It is not known whether γ is a rational or irrational number. □

2. Infinite series of real numbers

2.1. Definition and basic examples. The most important application of sequences is the definition of convergence of an infinite series. From the elementary school you have been dealing with addition of numbers. As you know the addition gets harder as you add more and more numbers. For example it would take some time to add

$$S_{100} = 1 + 2 + 3 + 4 + 5 + \cdots + 98 + 99 + 100$$

It gets much easier if you add two of these sums, and pair the numbers in a special way:

$$\begin{aligned} 2\,S_{100} = \quad & 1 + \ \ 2 + \ \ 3 + \ \ 4 + \cdots + 97 + 98 + 99 + 100 \\ & 100 + 99 + 98 + 97 + \cdots + \ \ 4 + \ \ 3 + \ \ 2 + \ \ \ 1 \end{aligned}$$

A straightforward observation that each column on the right adds to 101 and that there are 100 such columns yields that

$$2\,S_{100} = 101 \cdot 100, \qquad \text{that is} \qquad S_{100} = \frac{101 \cdot 100}{2} = 5050.$$

This can be generalized to any natural number n to get the formula

$$S_n = 1 + 2 + 3 + 4 + 5 + \cdots + (n-1) + n = \frac{(n+1)\,n}{2}.$$

This procedure indicates that it would be impossible to find the sum

$$1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots$$

where the last set of $\cdots$ indicates that we continue to add natural numbers.

The situation is quite different if we consider the sequence

$$\frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \ldots,\ \frac{1}{2^n},\ \ldots$$

and start adding more and more consecutive terms of this sequence:

$$\begin{aligned}
&\frac{1}{2} && = 1 - \frac{1}{2} = \frac{1}{2} \\
&\frac{1}{2} + \frac{1}{4} && = 1 - \frac{1}{4} = \frac{3}{4} \\
&\frac{1}{2} + \frac{1}{4} + \frac{1}{8} && = 1 - \frac{1}{8} = \frac{7}{8} \\
&\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} && = 1 - \frac{1}{16} = \frac{15}{16} \\
&\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} && = 1 - \frac{1}{32} = \frac{31}{32} \\
&\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} && = 1 - \frac{1}{64} = \frac{63}{64}
\end{aligned}$$

These sums are nicely illustrated in Fig. 1. The pictures in Fig. 1 strongly indicate that the sum of infinitely many numbers $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ equals 1. That is

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1$$

FIG. 1. In this example it seems natural to say that the sum of infinitely many numbers $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$ equals 1

Why does this make sense? This makes sense since we have seen above that as we add more and more terms of the sequence

$$\frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \dots,\ \frac{1}{2^n},\ \dots$$

we are getting closer and closer to 1. Indeed,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

and

$$\lim_{n\to+\infty}\left(1 - \frac{1}{2^n}\right) = 1.$$

This reasoning leads to the definition of convergence of an infinite series:

DEFINITION 2.1. Let $\{a_n\}_{n=1}^{+\infty}$ be a given sequence. Then the expression

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is called an infinite series. We often abbreviate it by writing

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{+\infty} a_n.$$

For each natural number n we calculate the (finite) sum of the first n terms of the series

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n.$$

We call S_n a partial sum of the infinite series $\sum_{n=1}^{+\infty} a_n$. (Notice that $\{S_n\}_{n=1}^{+\infty}$ is a new sequence.) If the sequence $\{S_n\}_{n=1}^{+\infty}$ converges to a real number S, that is if

$$\lim_{n\to+\infty} S_n = S,$$

then the infinite series $\sum_{n=1}^{+\infty} a_n$ is said to be *convergent* and we write

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = S \quad \text{or} \quad \sum_{n=1}^{+\infty} a_n = S.$$

The number S is called the *sum of the series*.

If the sequence $\{S_n\}_{n=1}^{+\infty}$ does not converge to a real number, then the series is called divergent.

In the example above we have

$$a_n = \frac{1}{2^n} = \left(\frac{1}{2}\right)^n,$$
$$S_n = 1 - \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$
$$\lim_{n\to+\infty} \left(1 - \frac{1}{2^n}\right) = 1.$$

Therefore we say that the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = \sum_{n=1}^{+\infty} \frac{1}{2^n}$$

converges and its sum is 1. We write $\sum_{n=1}^{+\infty} \frac{1}{2^n} = 1$.

In our opening example

$$a_n = n,$$
$$S_n = 1 + 2 + 3 + \cdots + n = \frac{(n+1)n}{2}$$
$$\lim_{n\to+\infty} \frac{(n+1)n}{2} \quad \text{does not exist.}$$

Therefore we say that the series

$$1 + 2 + 3 + 4 + \cdots + n + \cdots = \sum_{n=1}^{+\infty} n$$

diverges.

2.2. Geometric Series. Let a and r be real numbers. The most important infinite series is

$$a + a\,r + a\,r^2 + a\,r^3 + \cdots + a\,r^n + \cdots = \sum_{n=0}^{+\infty} a\,r^n \tag{2.1}$$

This series is called a geometric series. To determine whether this series converges or not we need to study its partial sums:

$$\begin{aligned}
&S_0 = a, && S_1 = a + a\,r,\\
&S_2 = a + a\,r + a\,r^2, && S_3 = a + a\,r + a\,r^2 + a\,r^3,\\
&S_4 = a + a\,r + a\,r^2 + a\,r^3 + a\,r^4, && S_5 = a + a\,r + a\,r^2 + a\,r^3 + a\,r^4 + a\,r^5,\\
&\ \ \vdots\\
&S_n = a + a\,r + a\,r^2 + \cdots + a\,r^{n-1} + a\,r^n\\
&\ \ \vdots
\end{aligned}$$

Notice that we have already studied the special case when $a = 1$ and $r = \dfrac{1}{2}$. In this special case we found a simple formula for S_n and then we evaluated $\lim\limits_{n\to+\infty} S_n$. It turns out that we can find a simple formula for S_n in the general case as well.

First note that the case $a = 0$ is not interesting, since then all the terms of the geometric series are equal to 0 and the series clearly converges and its sum is 0. Assume that $a \neq 0$. If $r = 1$ then $S_n = n\,a$. Since we assume that $a \neq 0$, $\lim\limits_{n\to+\infty} na$ does not exist. Thus for $r = 1$ the series diverges.

Assume that $r \neq 1$. To find a simple formula for S_n, multiply the long formula for S_n above by r to get:

$$\begin{aligned}
S_n &= a + ar + ar^2 + \cdots + ar^{n-1} + ar^n,\\
rS_n &= ar + ar^2 + \cdots + ar^n + ar^{n+1};
\end{aligned}$$

now subtract,

$$S_n - r\,S_n = a - a\,r^{n+1},$$

and solve for S_n:

$$S_n = a\,\frac{1 - r^{n+1}}{1 - r}\,.$$

We already proved that if $|r| < 1$, then $\lim\limits_{n\to+\infty} r^{n+1} = 0$. If $|r| \geq 1$, then $\lim\limits_{n\to+\infty} r^{n+1}$ does not exist. Therefore we conclude that

$$\begin{aligned}
&\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} a\,\frac{1 - r^{n+1}}{1 - r} = a\,\frac{1}{1 - r} && \text{for} \quad |r| < 1,\\
&\lim_{n\to+\infty} S_n \quad \text{does not converge to a real number} && \text{for} \quad |r| \geq 1.
\end{aligned}$$

In conclusion

- If $|r| < 1$, then the geometric series $\displaystyle\sum_{n=0}^{+\infty} a\,r^n$ converges and its sum is $a\,\dfrac{1}{1 - r}$.
- If $|r| \geq 1$, then the geometric series $\displaystyle\sum_{n=0}^{+\infty} a\,r^n$ diverges.

Fig. 2 illustrates the sum of a geometric series with $a > 0$ and $0 < r < 1$:

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \frac{a}{1-r}.$$

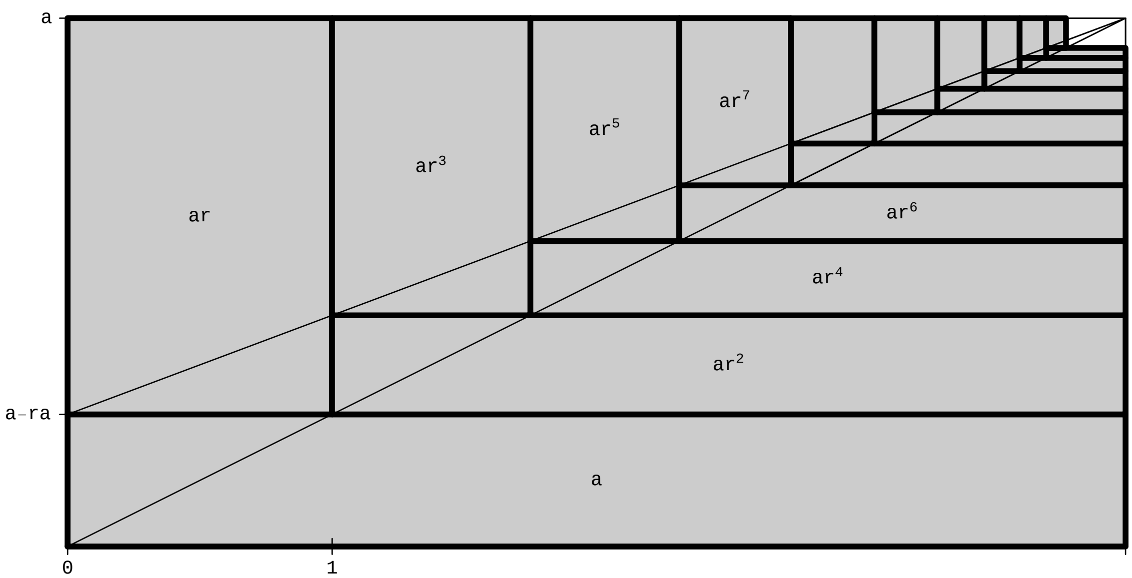

FIG. 2. The width of the rectangle is $1/(1-r)$ and the height is a. The slopes of the lines shown are $(1-r)a$ and $r(1-r)a$

In Fig. 2 the terms of a geometric series are represented as areas. As we can see in Fig. 2 the areas of the terms fill in the rectangle whose area is $a/(1-r)$.

In Fig. 3 we represent the terms of the geometric series by lengths of horizontal line segments. The picture strongly indicates that the total length of infinitely many horizontal line segments is $a/(1-r)$. The reason for this is that by the construction the slope of the hypothenuse of the right triangle in Fig. 3 is $(1-r)$. Since its vertical leg is a, its horizontal leg must be $a/(1-r)$.

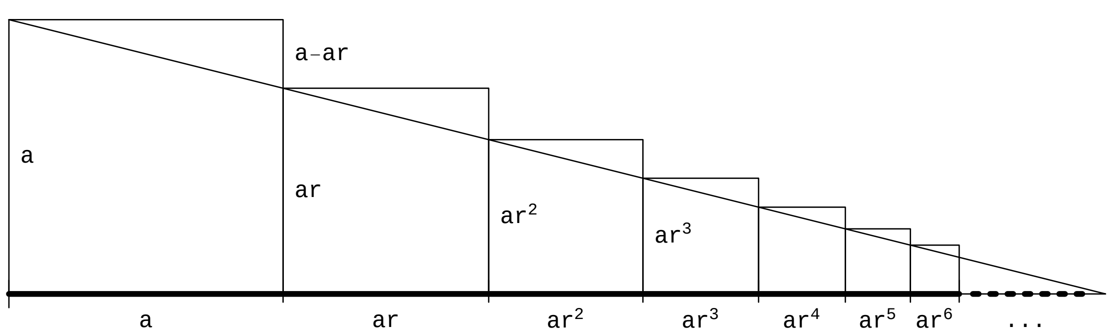

FIG. 3. The slope of the hypothenuse is $1-r = \frac{\text{rise}}{\text{run}} = \frac{a-ar}{a}$ (see the topmost right triangle). This implies that the horizontal leg of the largest right triangle is $a/(1-r)$: $1-r = \frac{\text{rise}}{\text{run}} = \frac{a}{\frac{a}{1-r}}$.

2.3. How to recognize whether an infinite series is a geometric series? Consider for example the infinite series $\displaystyle\sum_{n=1}^{+\infty} \frac{\pi^{n+2}}{e^{2n-1}}$. Here $a_n = \dfrac{\pi^{n+2}}{e^{2n-1}}$.

Looking at the formula (2.1) we note that the first term of the series is a and that the ratio between any two consecutive terms is r.

For $a_n = \dfrac{\pi^{n+2}}{e^{2n-1}}$ given above we calculate

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{\pi^{n+1+2}}{e^{2(n+1)-1}}}{\dfrac{\pi^{n+2}}{e^{2n-1}}} = \frac{\pi^{n+3}\, e^{2n-1}}{e^{2n+1}\, \pi^{n+2}} = \frac{\pi}{e^2}\,.$$

Since $\dfrac{a_{n+1}}{a_n}$ is constant, we conclude that the series $\displaystyle\sum_{n=1}^{+\infty} \frac{\pi^{n+2}}{e^{2n-1}}$ is a geometric series with

$$a = a_1 = \frac{\pi^2}{e} \quad \text{and} \quad r = \frac{\pi}{e^2} \quad \text{for all} \quad n = 1, 2, 3, \dots\,.$$

Since $r = \dfrac{\pi}{e^2} < 1$, we conclude that the sum of this series is

$$\sum_{n=1}^{+\infty} \frac{\pi^{n+2}}{e^{2n-1}} = \frac{\pi^2}{e}\,\frac{1}{1-\dfrac{\pi}{e^2}} = \frac{\pi^2}{e}\,\frac{e^2}{e^2-\pi} = \frac{\pi^2\, e}{e^2-\pi}\,.$$

Thus, to verify whether a given infinite series is a geometric series calculate the ratio of the consecutive terms and see whether it is a constant:

$$\sum_{n=1}^{+\infty} a_n \quad \text{for which} \quad \frac{a_{n+1}}{a_n} = r \quad \text{for all} \quad n = 1, 2, 3, \dots \tag{2.2}$$

is a geometric series. In this case $a = a_1$ (the first term of the series).

2.4. Harmonic Series. Harmonic series is the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots = \sum_{n=1}^{+\infty} \frac{1}{n}\,.$$

Again, to explore the convergence of this series we have to study its partial sums:

$$\begin{aligned}
&S_1 = 1, && S_2 = 1 + \frac{1}{2},\\
&S_3 = 1 + \frac{1}{2} + \frac{1}{3}, && S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4},\\
&S_5 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}, && S_6 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6},\\
&S_7 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}, && S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8},\\
&\quad\vdots\\
&S_n == 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}
\end{aligned}$$

$\vdots$

Since $S_{n+1} - S_n = \dfrac{1}{n+1} > 0$ the sequence $\{S_n\}_{n=1}^{+\infty}$ is increasing.

Next we will prove that the sequence $\{S_n\}_{n=1}^{+\infty}$ is not bounded. We will consider only the natural numbers which are powers of 2: $2, 4, 8, \ldots, 2^k, \ldots$. The following inequalities hold:

$$S_2 = 1 + \frac{1}{2} \geq 1 + \frac{1}{2} \qquad = 1 + 1\,\frac{1}{2}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + 2\,\frac{1}{4} \qquad = 1 + 2\,\frac{1}{2}$$

$$\begin{aligned} S_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \\ &\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 1 + \frac{1}{2} + 2\,\frac{1}{4} + 4\,\frac{1}{8} \qquad = 1 + 3\,\frac{1}{2} \end{aligned}$$

$$\begin{aligned} S_{16} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} \\ &\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 1 + \frac{1}{2} + 2\,\frac{1}{4} + 4\,\frac{1}{8} + 8\,\frac{1}{16} \qquad = 1 + 4\,\frac{1}{2} \end{aligned}$$

Continuing this reasoning we conclude that for each $k = 1, 2, 3, \ldots$ the following formula holds:

$$\begin{aligned} S_{2^k} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{8} + \cdots + \frac{1}{2^{k-1}} + \frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k} \\ &\geq 1 + \frac{1}{2} + 2\,\frac{1}{4} + 4\,\frac{1}{8} + 8\,\frac{1}{16} + \cdots + 2^{k-1}\,\frac{1}{2^k} \qquad = 1 + k\,\frac{1}{2} \end{aligned}$$

Thus

$$S_{2^k} \geq 1 + k\,\frac{1}{2} \quad \text{for all} \quad k = 1, 2, 3, \ldots \tag{2.3}$$

This formula implies that the sequence $\{S_n\}_{n=1}^{+\infty}$ is not bounded. Namely, let M be an arbitrary real number. We put $j = \max\bigl\{2\,\mathrm{floor}(M), 1\bigr\}$. Then

$$j \geq 2\,\mathrm{floor}(M) > 2(M-1).$$

Therefore,

$$1 \,+\, j\,\frac{1}{2} > M.$$

Together with the inequality (2.3) this implies that

$$S_{2^j} > M.$$

Thus for an arbitrary real number M there exists a natural number $n = 2^j$ such that $S_n > M$. This proves that the sequence $\{S_n\}_{n=1}^{+\infty}$ is not bounded and therefore it is not convergent.

In conclusion:

- The harmonic series diverges.

2.5. Telescoping series. The next example is an example of a series for which we can find a simple formula for the sequence of its partial sums and easily explore the convergence of that sequence. Examples of this kind are called telescoping series.

EXAMPLE 2.2. Prove that the series $\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$ converges and find its sum.

SOLUTION. We need to examine the series of partial sums of this series:

$$S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}, \quad n = 1, 2, 3, \dots .$$

It turns out that it is easy to find the sum S_n if we use the partial fraction decomposition for each of the terms of the series:

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} \quad \text{for all} \quad k = 1, 2, 3, \dots .$$

Now we calculate:

$$\begin{aligned} S_n &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\,(n+1)} \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}\,. \end{aligned}$$

Thus $S_n = 1 - \frac{1}{n+1}$ for all $n = 1, 2, 3, \dots$. Using the algebra of limits we conclude that

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \left(1 - \frac{1}{n+1}\right) = 1\,.$$

Therefore the series $\sum_{n=1}^{+\infty} \frac{1}{n\,(n+1)}$ converges and its sum is 1:

$$\sum_{n=1}^{+\infty} \frac{1}{n\,(n+1)} = 1\,.$$

□

EXERCISE 2.3. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

(a) $\sum_{n=1}^{+\infty} 6\left(\frac{2}{3}\right)^{n-1}$ (b) $\sum_{n=1}^{+\infty} \frac{(-2)^{n+3}}{5^{n-1}}$ (c) $\sum_{n=0}^{+\infty} \frac{(\sqrt{2})^n}{2^{n+1}}$ (d) $\sum_{n=1}^{+\infty} \frac{e^{n+3}}{\pi^{n-1}}$

(e) $\sum_{n=1}^{+\infty} \frac{2^{2n-1}}{\pi^n}$ (f) $\sum_{n=1}^{+\infty} \frac{5}{2\,n}$ (g) $\sum_{n=0}^{+\infty} (\sin 1)^n$ (h) $\sum_{n=0}^{+\infty} \frac{2}{n^2+4n+3}$

(i) $\sum_{n=0}^{+\infty} (\cos 1)^n$ (j) $\sum_{n=2}^{+\infty} \frac{2}{n^2-1}$ (k) $\sum_{n=0}^{+\infty} (\tan 1)^n$ (l) $\sum_{n=1}^{+\infty} \ln\left(1 + \frac{1}{n}\right)$

A digit is a number from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. A decimal number with digits $d_1, d_2, d_3, \dots, d_n, \dots$ is in fact an infinite series:

$$0.d_1d_2d_3\dots d_n\dots = \sum_{n=1}^{+\infty} \frac{d_n}{10^n}.$$

Therefore each decimal number with digits that repeat leads to a geometric series. We use the following abbreviation:

$$0.\overline{d_1d_2d_3\dots d_k} = 0.d_1d_2d_3\dots d_kd_1d_2d_3\dots d_kd_1d_2d_3\dots d_kd_1d_2d_3\dots d_k\dots$$

Exercise 2.4. Express the number as a ratio of integers.
(a) $0.\overline{9} = 0.999\dots$ (b) $0.\overline{7} = 0.777\dots$ (c) $0.\overline{712}$ (d) $0.\overline{5432}$

2.6. Basic properties of infinite series. An immediate consequence of the definition of a convergent series is the following theorem

Theorem 2.5. *If a series* $\sum_{n=1}^{+\infty} a_n$ *converges, then* $\lim_{n\to+\infty} a_n = 0$.

Proof. Assume that $\sum_{n=1}^{+\infty} a_n$ is a convergent series. By the definition of convergence of a series its sequence of partial sums $\{S_n\}_{n=1}^{+\infty}$ converges to some number S: $\lim_{n\to+\infty} S_n = S$. Then also $\lim_{n\to+\infty} S_{n-1} = S$. Now using the formula

$$a_n = S_n - S_{n-1}, \quad \text{for all} \quad n = 2, 3, 4, \dots,$$

and the algebra of limits we conclude that

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} S_n - \lim_{n\to+\infty} S_{n-1} = S - S = 0.$$

□

Warning: The preceding theorem cannot be used to conclude that a particular series converges. Notice that in this theorem it is <u>a</u>ssumed that $\sum_{n=1}^{+\infty} a_n$ is a convergent.

On a positive note: Theorem 2.5 can be used to conclude that a given series diverges: If we know that $\lim_{n\to+\infty} a_n = 0$ is not true, then we can conclude that the series $\sum_{n=1}^{+\infty} a_n$ diverges. This is a useful test for divergence.

Theorem 2.6 (The Test for Divergence). *If the sequence* $\{a_n\}_{n=1}^{+\infty}$ *does not converge to* 0, *then the series* $\sum_{n=1}^{+\infty} a_n$ *diverges.*

Example 2.7. Determine whether the infinite series $\sum_{n=1}^{+\infty} \cos\left(\frac{1}{n}\right)$ converges or diverges.

SOLUTION. Just perform the divergence test:

$$\lim_{n\to+\infty} \cos\left(\frac{1}{n}\right) = 1 \neq 0\,.$$

Therefore the series $\sum_{n=1}^{+\infty} \cos\left(\frac{1}{n}\right)$ diverges. □

EXAMPLE 2.8. Determine whether the infinite series $\sum_{n=1}^{+\infty} \frac{n^{(-1)^n}}{n+1}$ converges or diverges.

SOLUTION. Consider the sequence $\left\{\frac{n^{(-1)^n}}{n+1}\right\}_{n=1}^{+\infty}$:

$$\frac{1}{1\cdot 2}, \frac{2}{3}, \frac{1}{3\cdot 4}, \frac{4}{5}, \frac{1}{5\cdot 6}, \frac{6}{7}, \frac{1}{7\cdot 8}, \frac{8}{9}, \frac{1}{9\cdot 10}, \frac{10}{11}, \frac{1}{11\cdot 12}, \frac{12}{13}, \ldots, \frac{1}{(2k-1)\cdot 2k}, \frac{2k}{2k+1}, \ldots \tag{2.4}$$

Without giving a formal proof we can tell that this sequence diverges. In my informal language the sequence (2.4) is not constantish since it can not decide whether to be close to 0 or 1.

Therefore the series $\sum_{n=1}^{+\infty} \frac{n^{(-1)^n}}{n+1}$ diverges. □

REMARK 2.9. The divergence test can not be used to answer whether the series $\sum_{n=1}^{+\infty} \sin\left(\frac{1}{n}\right)$ converges or diverges. It is clear that $\lim_{n\to+\infty} \sin\left(\frac{1}{n}\right) = 0$. Thus we can not use the test for divergence.

THEOREM 2.10 (The Algebra of Convergent Infinite Series). *Assume that* $\sum_{n=1}^{+\infty} a_n$ *and* $\sum_{n=1}^{+\infty} b_n$ *are convergent series. Let* c *be a real number. Then the series*

$$\sum_{n=1}^{+\infty} c\,a_n, \qquad \sum_{n=1}^{+\infty}(a_n+b_n), \qquad \textit{and} \qquad \sum_{n=1}^{+\infty}(a_n-b_n),$$

are convergent series and the following formulas hold

$$\sum_{n=1}^{+\infty} c\,a_n = c\sum_{n=1}^{+\infty} a_n,$$

$$\sum_{n=1}^{+\infty}(a_n+b_n) = \sum_{n=1}^{+\infty} a_n + \sum_{n=1}^{+\infty} b_n, \quad \textit{and}$$

$$\sum_{n=1}^{+\infty}(a_n-b_n) = \sum_{n=1}^{+\infty} a_n - \sum_{n=1}^{+\infty} b_n\,.$$

REMARK 2.11. The fact that we write $\sum_{n=1}^{+\infty} b_n$ does not necessarily mean that $\sum_{n=1}^{+\infty} b_n$ is a genuine infinite series.

For example, let m be a natural number and assume that $b_n = 0$ for all $n > m$. Then $\sum_{n=1}^{+\infty} b_n = \sum_{n=1}^{m} b_n$. In this case the series $\sum_{n=1}^{+\infty} b_n$ is clearly convergent. If $\sum_{n=1}^{+\infty} a_n$ is a convergent (genuine) infinite series, then Theorem 2.10 implies that the infinite series $\sum_{n=1}^{+\infty} (a_n + b_n)$ is convergent and

$$\sum_{n=1}^{+\infty} (a_n + b_n) = \sum_{n=1}^{+\infty} a_n + \sum_{n=1}^{m} b_n .$$

This in particular means that the nature of convergence of an infinite series can not be changed by changing finitely many terms of the series.

For example, let m be a natural number. Then:

The series $\displaystyle\sum_{n=1}^{+\infty} a_n$ converges if and only if the series $\displaystyle\sum_{k=1}^{+\infty} a_{m+k}$ converges.

Moreover, if $\sum_{n=1}^{+\infty} a_n$ converges, then the following formula holds

$$\sum_{n=1}^{+\infty} a_n = \sum_{j=1}^{m} a_j + \sum_{k=1}^{+\infty} a_{m+k} .$$

EXAMPLE 2.12. Prove that the series $\displaystyle\sum_{n=1}^{+\infty} \left(\frac{\pi}{n(n+1)} - \frac{1}{2^n} \right)$ converges and find its sum.

EXERCISE 2.13. Determine whether the series is convergent or divergent. If a series is convergent find its sum.

(a) $\displaystyle\sum_{n=1}^{+\infty} \frac{n}{n+1}$ (b) $\displaystyle\sum_{n=1}^{+\infty} \arctan n$ (c) $\displaystyle\sum_{n=0}^{+\infty} \frac{3^n + 2^n}{5^{n+1}}$ (d) $\displaystyle\sum_{n=2}^{+\infty} \left(\frac{3}{n^2 - 1} + \frac{\pi}{e^n} \right)$

(e) $\displaystyle\sum_{n=0}^{+\infty} \frac{e^n + \pi^n}{2^{2n-1}}$ (f) $\displaystyle\sum_{n=1}^{+\infty} n \sin\left(\frac{1}{n}\right)$ (g) $\displaystyle\sum_{n=0}^{+\infty} \frac{(n+1)^2}{n^2+1}$ (h) $\displaystyle\sum_{n=0}^{+\infty} \left((0.9)^n + (0.1)^n\right)$

EXERCISE 2.14. Express the following sums as ratios of integers and as repeating decimal numbers.

(a) $0.\overline{47} + 0.\overline{5}$ (b) $0.\overline{499} + 0.\overline{47}$ (c) $0.\overline{499} + 0.\overline{503}$

3. Convergence Tests

3.1. Comparison Theorems. Warning: All series in the next two subsections have positive terms! Do not use the tests from these sections for series with some negative terms.

The convergence of the geometric series in Subsection 2.2 and the telescopic series in Subsection 2.5 was established by calculating the limits of their partial sums. This is not possible for most series. For example we will soon prove that the series

$$\sum_{n=1}^{+\infty} \frac{1}{n^2}$$

converges. To understand why the sum of this series is exactly $\dfrac{\pi^2}{6}$ you need to take a class about Fourier series, Math 430.

I hope that you have done your homework and that you proved that the series

$$\sum_{n=2}^{+\infty} \frac{1}{n^2-1}$$

converges and that you found its sum. If you didn't here is a way to do it: (It turns out that this is a telescoping series.)

Let

$$S_n = \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \cdots + \frac{1}{n^2-1}.$$

Since $S_{n+1} - S_n = \dfrac{1}{(n+1)^2-1} > 0$ the sequence $\{S_n\}_{n=2}^{+\infty}$ is increasing.

For each $k = 2, 3, 4, \ldots$ we have the following partial fractions decomposition

$$\frac{1}{k^2-1} = \frac{1}{(k-1)(k+1)} = \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right).$$

Next we use this formula to simplify the formula for the n-th partial sum

$$\begin{aligned}
S_n &= \sum_{k=2}^{n} \frac{1}{k^2-1} = \sum_{k=2}^{n} \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right) = \frac{1}{2}\sum_{k=2}^{n}\left(\frac{1}{k-1} - \frac{1}{k+1}\right) \\
&= \frac{1}{2}\left(\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-2} - \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{1}{n+1}\right)\right) \\
&= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right) \\
&= \frac{1}{2}\left(\frac{3}{2} - \frac{2n+1}{n(n+1)}\right) = \frac{3}{4} - \frac{2n+1}{2n(n+1)}.
\end{aligned}$$

Using the algebra of limits we calculate

$$\lim_{n\to+\infty} \frac{2n+1}{2n(n+1)} = \lim_{n\to+\infty} \frac{\dfrac{2n+1}{n^2}}{\dfrac{2n(n+1)}{n^2}} = \lim_{n\to+\infty} \frac{\dfrac{2}{n} + \dfrac{1}{n^2}}{2\dfrac{n+1}{n}} = \frac{0+0}{2\cdot 1} = 0.$$

Therefore, using the algebra of limits again, we calculate

$$\lim_{n\to+\infty} S_n = \frac{3}{4} - 0 = \frac{3}{4}\,.$$

Clearly $S_n < \dfrac{3}{4}$ for all $n = 2, 3, \ldots$.

Now consider the series

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$$

Let

$$T_n = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2}\,.$$

The fact that $T_{n+1} - T_n = \dfrac{1}{(n+1)^2} > 0$ implies that the sequence $\{T_n\}_{n=1}^{+\infty}$ is increasing.

Since

$$\frac{1}{4} < \frac{1}{3}, \quad \frac{1}{9} < \frac{1}{8}, \quad \frac{1}{16} < \frac{1}{15}, \quad \ldots, \quad \frac{1}{n^2} < \frac{1}{n^2-1},$$

we conclude that

$$T_n = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} < 1 + \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \cdots + \frac{1}{n^2-1} = 1 + S_n < 1 + \frac{3}{4}\,.$$

Thus $T_n < \dfrac{7}{4}$ for all $n = 2, 3, 4, \ldots$. Since the sequence $\{T_n\}_{n=1}^{+\infty}$ is increasing and bounded above it converges by Theorem 1.16. Thus the series $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2}$ converges and its sum is $< \dfrac{7}{4}$.

The principle demonstrated in the above example is the core of the following comparison theorem.

THEOREM 3.1 (The Comparison Test). *Let* $\displaystyle\sum_{n=1}^{+\infty} a_n$ *and* $\displaystyle\sum_{n=1}^{+\infty} b_n$ *be infinite series with positive terms. Assume that*

$$a_n \le b_n \qquad \textit{for all} \qquad n = 1, 2, 3, \ldots\,.$$

(a) *If* $\displaystyle\sum_{n=1}^{+\infty} b_n$ *converges, then* $\displaystyle\sum_{n=1}^{+\infty} a_n$ *converges and* $\displaystyle\sum_{n=1}^{+\infty} a_n \le \sum_{n=1}^{+\infty} b_n$.

(b) *If* $\displaystyle\sum_{n=1}^{+\infty} a_n$ *diverges, then* $\displaystyle\sum_{n=1}^{+\infty} b_n$ *diverges.*

Sometimes the following comparison theorem is easier to use.

THEOREM 3.2 (The Limit Comparison Test). *Let* $\displaystyle\sum_{n=1}^{+\infty} a_n$ *and* $\displaystyle\sum_{n=1}^{+\infty} b_n$ *be infinite series with positive terms. Assume that*

$$\lim_{n\to+\infty} \frac{a_n}{b_n} = L.$$

If $\displaystyle\sum_{n=1}^{+\infty} b_n$ *converges, then* $\displaystyle\sum_{n=1}^{+\infty} a_n$ *converges. Or, equivalently, if* $\displaystyle\sum_{n=1}^{+\infty} a_n$ *diverges, then* $\displaystyle\sum_{n=1}^{+\infty} b_n$ *diverges.*

Example 3.3. Determine whether the series $\sum_{n=1}^{+\infty} \frac{n+1}{\sqrt{1+n^6}}$ converges or diverges.

Solution. The dominant term in the numerator is n and the dominant term in the denominator is $\sqrt{n^6} = n^3$. This suggests that this series behaves as the convergent series $\sum_{n=1}^{+\infty} \frac{1}{n^2}$. Since we are trying to prove convergence we will take

$$a_n = \frac{n+1}{\sqrt{1+n^6}} \quad \text{and} \quad b_n = \frac{1}{n^2}$$

in the Limit Comparison Test. Now calculate:

$$\lim_{n\to+\infty} \frac{\dfrac{n+1}{\sqrt{1+n^6}}}{\dfrac{1}{n^2}} = \lim_{n\to+\infty} \frac{n^2(n+1)}{\sqrt{1+n^6}} = \lim_{n\to+\infty} \frac{\dfrac{n^2(n+1)}{n^3}}{\dfrac{\sqrt{1+n^6}}{n^3}} = \lim_{n\to+\infty} \frac{1+\dfrac{1}{n}}{\sqrt{\dfrac{1}{n^6}+1}} = 1\,.$$

In the last step we used the algebra of limits and the fact that

$$\lim_{n\to+\infty} \sqrt{\frac{1}{n^6}+1} = 1$$

which needs a proof by definition.

Since we proved that $\lim_{n\to+\infty} \frac{\frac{n+1}{\sqrt{1+n^6}}}{\frac{1}{n^2}} = 1$ and since we know that $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ is convergent, the Limit Comparison Test implies that the series $\sum_{n=1}^{+\infty} \frac{n+1}{\sqrt{1+n^6}}$ converges. □

In the next theorem we compare an infinite series with an improper integral of a positive function. Here it is presumed that we know how to determine the convergence or divergence of the improper integral involved.

Theorem 3.4 (The Integral Test). *Suppose that $x \mapsto f(x)$ is a continuous positive, decreasing function defined on the interval $(0,+\infty)$. Assume that $a_n = f(n)$ for all $n = 1, 2, \ldots$. Then the following statements are equivalent*

(a) *The integral $\int_1^{+\infty} f(x)\,dx$ converges.*

(b) *The series $\sum_{n=1}^{+\infty} a_n$ converges.*

At this point we assume that you are familiar with improper integrals and that you know how to decide whether an improper integral converges or diverges.

We will use this test in two different forms:

- Prove that the integral $\int_1^{+\infty} f(x)\,dx$ converges. Conclude that the series $\sum_{n=1}^{+\infty} a_n$ converges.

- Prove that the integral $\int_1^{+\infty} f(x)\,dx$ diverges. Conclude that the series $\sum_{n=1}^{+\infty} a_n$ diverges.

EXAMPLE 3.5 (Convergence of p-series). Let p be a real number. The p-series $\sum_{n=1}^{+\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

SOLUTION. Let $n > 1$. Then the function $x \mapsto n^x$ is an increasing function. Therefore, if $p < 1$, then $n^p < n$. Consequently,

$$\frac{1}{n^p} > \frac{1}{n}, \quad \text{for all} \quad n > 1 \quad \text{and} \quad p < 1\,.$$

Since the series $\sum_{n=1}^{+\infty} \frac{1}{n}$ diverges, the Comparison Test implies that the series $\sum_{n=1}^{+\infty} \frac{1}{n^p}$ diverges for all $p \leq 1$.

Now assume that $p > 1$. Consider the function $f(x) = \frac{1}{x^p}$, $x > 0$. This function is a continuous, decreasing, positive function. Let me calculate the improper integral involved in the Integral Test for convergence:

$$\begin{aligned}
\int_1^{+\infty} \frac{1}{x^p}\,dx &= \lim_{t\to+\infty} \int_1^t \frac{1}{x^p}\,dx = \lim_{t\to+\infty} \frac{1}{1-p}\,\frac{1}{x^{p-1}}\,\Bigg|_1^t \\
&= \frac{1}{1-p} \lim_{t\to+\infty} \left(\frac{1}{t^{p-1}} - 1 \right) = \frac{1}{1-p}(-1) = \frac{1}{p-1}
\end{aligned}$$

Thus this improper integral converges. Notice that the condition $p > 1$ was essential to conclude that $\lim_{t\to+\infty} \frac{1}{t^{p-1}} = 0$. Since $\frac{1}{n^p} = f(n)$ for all $n = 1, 2, 3, \ldots$, the Integral Test implies that the series $\sum_{n=1}^{+\infty} \frac{1}{n^p}$ converges for $p > 1$. □

REMARK 3.6. We have not proved this for all $p > 1$ the function $f(x) = \frac{1}{x^p}$, $x > 0$, is continuous. One way to prove that for an arbitrary $a \in \mathbb{R}$ the function $x \mapsto x^a$, $x > 0$ is continuous is to use the identity

$$x^a = e^{a \ln x}, \qquad x > 0.$$

This identity shows that the function $x \mapsto x^a$, $x > 0$ is a composition of the function $\exp(x) = e^x$, $x \in \mathbb{R}$ and the function $x \mapsto a\,\ln x$, $x > 0$. The later function is continuous by the algebra of continuous functions: It is a product of a constant a and a continuous function ln. We proved that exp is continuous. By Theorem 6.16 a composition of continuous function is continuous. Consequently $x \mapsto x^a$, $x > 0$ is continuous.

EXERCISE 3.7. Determine whether the series is convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n\sqrt{n}}$ (b) $\displaystyle\sum_{n=1}^{+\infty} ne^{-n^2}$ (c) $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{n\ln n}$ (d) $\displaystyle\sum_{n=1}^{+\infty} \frac{\ln n}{n\sqrt{n}}$

(e) $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{n(\ln n)^b}$ (f) $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n!}$ (g) $\displaystyle\sum_{n=1}^{+\infty} \sin\big(\frac{1}{n}\big)$ (h) $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{n}\sin\big(\frac{1}{n}\big)$

(i) $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n}\cos\big(\frac{1}{n}\big)$ (j) $\displaystyle\sum_{n=0}^{+\infty} \frac{\pi+e^n}{e+\pi^n}$ (k) $\displaystyle\sum_{n=1}^{+\infty} \frac{n!}{n^n}$ (l) $\displaystyle\sum_{n=0}^{+\infty} \frac{n^2+1}{\sqrt{n^7+n^3+1}}$

For the series in (e) find all numbers b for which the series converges.

Exercise 3.8. A digit is a number from the set $\{0,1,2,3,4,5,6,7,8,9\}$. A decimal number with digits $d_1, d_2, d_3, \ldots, d_n, \ldots$ is in fact an infinite series:

$$0.d_1d_2d_3\ldots d_n\ldots \;=\; \sum_{n=1}^{+\infty} \frac{d_n}{10^n}\,.$$

Use a theorem from this section to prove that the series above always converges.

3.2. Ratio and root tests. Warning: All series in this section have positive terms! Do not use the tests from this section for series with negative terms.

In Subsection 2.3 we pointed out (see (2.2)) that a series

$$\sum_{n=1}^{+\infty} a_n \quad \text{for which} \quad \frac{a_{n+1}}{a_n} = r \quad \text{for all} \quad n = 1,2,3,\ldots$$

is a **geometric series**. Consequently, if $|r| < 1$ this series is convergent, and it is divergent if $|r| \geq 1$.

Testing the series $\displaystyle\sum_{n=0}^{+\infty} \frac{1}{3^n - 2^{n+1}}$ using this criteria leads to the ratio

$$\frac{\dfrac{1}{3^{n+1}-2^{n+2}}}{\dfrac{1}{3^n-2^{n+1}}} = \frac{3^n-2^{n+1}}{3^{n+1}-2^{n+2}} = \frac{3^n\left(1-2\left(\dfrac{2}{3}\right)^n\right)}{3^{n+1}\left(1-2\left(\dfrac{2}{3}\right)^n\right)} = \frac{1}{3}\;\frac{1-2\left(\dfrac{2}{3}\right)^n}{1-2\left(\dfrac{2}{3}\right)^{n+1}}$$

which certainly is not constant, but it is "constantish." I propose that series for which the ratio a_{n+1}/a_n is not constant but constantish, should be called "geometrish." The following theorem tells that convergence and divergence of these series is determined similarly to geometric series.

Theorem 3.9 (The Ratio Test). *Assume that* $\displaystyle\sum_{n=1}^{+\infty} a_n$ *is a series with positive terms and that*

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = R\,.$$

Then

(a) *If* $R < 1$, *then the series converges.*
(b) *If* $R > 1$, *then the series diverges.*

Another way to recognize a geometric series is:

$$\text{A series} \quad \sum_{n=1}^{+\infty} a_n \quad \text{for which} \quad \sqrt[n]{\frac{a_{n+1}}{a_1}} = r \quad \text{for all} \quad n = 1, 2, 3, \ldots$$

is a **geometric series**. Consequently, if $|r| < 1$ this series is convergent, and it is divergent if $|r| \geq 1$.

Testing the series $\displaystyle\sum_{n=0}^{+\infty} \left(\frac{1+n}{1+2n}\right)^n$ using this criteria leads to the root

$$\sqrt[n]{\left(\frac{1+n}{1+2n}\right)^n} = \frac{1+n}{1+2n} = \frac{\frac{1}{n}+1}{\frac{1}{n}+2}$$

which certainly is not constant, but it is "constantish."

THEOREM 3.10 (The Root Test). *Assume that* $\displaystyle\sum_{n=1}^{+\infty} a_n$ *is a series with positive terms and that*

$$\lim_{n \to +\infty} \sqrt[n]{a_n} = R\,.$$

Then

(a) *If* $R < 1$, *then the series converges.*
(b) *If* $R > 1$, *then the series diverges.*

REMARK 3.11. Notice that in both the ratio test and the root test if the limit $R = 1$ we can conclude neither divergence nor convergence. In this case the test is inconclusive.

EXERCISE 3.12. Determine whether the series is convergent or divergent.

(a) $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{2^n - 3}$	(b) $\displaystyle\sum_{n=1}^{+\infty} \left(\frac{n+2}{2n-1}\right)^n$	(c) $\displaystyle\sum_{n=1}^{+\infty} \frac{4^n}{3^{2n-1}}$	(d) $\displaystyle\sum_{n=1}^{+\infty} \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$
(e) $\displaystyle\sum_{n=1}^{+\infty} \frac{3^n n^2}{n!}$	(f) $\displaystyle\sum_{n=1}^{+\infty} e^{-n} n!$	(g) $\displaystyle\sum_{n=1}^{+\infty} \frac{e^{1/n}}{n^2}$	(h) $\displaystyle\sum_{n=1}^{+\infty} \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$
(i) $\displaystyle\sum_{n=1}^{+\infty} \frac{(n!)^2}{(2n)!}$	(j) $\displaystyle\sum_{n=1}^{+\infty} \frac{2\,n^{2n}}{(3n^2+1)^n}$	(k) $\displaystyle\sum_{n=1}^{+\infty} \frac{2^{3n}}{3^{2n}}$	(l) $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{(\arctan n)^n}$
(m) $\displaystyle\sum_{n=1}^{+\infty} \frac{n^2}{2^n}$	(n) $\displaystyle\sum_{n=1}^{+\infty} \frac{(n+1)^2}{n 2^n}$	(o) $\displaystyle\sum_{n=1}^{+\infty} \frac{a^n}{n!}$	(p) $\displaystyle\sum_{n=1}^{+\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$

For some of the problems you might need to use tests from previous sections.

3.3. Alternating infinite series. In the previous two sections we considered only series with positive terms. In this section we consider series with both positive and negative terms which alternate: positive, negative, positive, etc. Such series are called **alternating series**. For example

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots+(-1)^{n+1}\frac{1}{n}+\cdots=\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{n} \tag{3.1}$$

$$1-1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}-\frac{1}{3}+\frac{1}{7}-\frac{1}{4}+\frac{1}{8}-\frac{1}{5}+\frac{1}{9}-\frac{1}{6}+\cdots=\sum_{n=1}^{+\infty}\frac{4(-1)^{n+1}}{n\big(3+(-1)^{n+1}\big)} \tag{3.2}$$

$$2-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\frac{6}{5}-\frac{7}{6}+\cdots+(-1)^{n+1}\frac{n+1}{n}+\cdots=\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{n+1}{n} \tag{3.3}$$

THEOREM 3.13 (The Alternating Series Test). *If the alternating series*

$$a_1-a_2+a_3-a_4+\cdots+(-1)^{n+1}a_n+\cdots=\sum_{n=1}^{+\infty}(-1)^{n+1}a_n$$

satisfies the following two conditions:

(i) $0\le a_{n+1}\le a_n$ *for all* $n=1,2,3,\dots,$

(ii) $\displaystyle\lim_{n\to+\infty}a_n=0,$

then the series is convergent.

PROOF. Assume that $\{a_n\}_{n=1}^{+\infty}$ satisfies (i) and (ii).

By the definition of convergence the assumption (ii) implies that for each $\epsilon>0$ there exists $N(\epsilon)$ such that

$$n\in\mathbb{N},\quad n>N(\epsilon)\qquad\Rightarrow\qquad |a_n-0|<\epsilon.$$

Since $a_n\ge 0$, the last implication can be simplified as

$$n\in\mathbb{N},\quad n>N(\epsilon)\quad\text{implies}\quad a_n<\epsilon. \tag{3.4}$$

We need to show that the sequence of partial sums $\{S_n\}_{n=1}^{+\infty}$,

$$S_n=a_1-a_1-a_2+a_3-a_4+\cdots+(-1)^{n+1}a_n,\qquad n\in\mathbb{N},$$

is convergent.

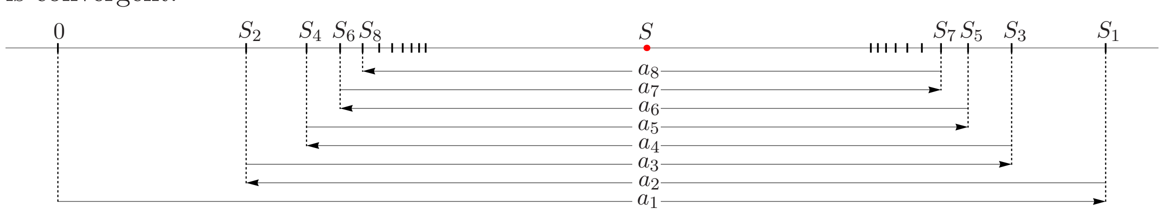

FIG. 4. The partial sums of an alternating series on a number line

First consider the sequence $\{S_{2n}\}_{n=1}^{+\infty}$ of even partial sums. Since $a_{2n+2}\le a_{2n+1}$, we have

$$S_{2(n+1)}-S_{2n}=a_{2n+1}-a_{2n+2}\ge 0.$$

Thus the sequence $\{S_{2n}\}_{n=1}^{+\infty}$ is non-decreasing.

Next we compare an arbitrary even partial sum S_{2k} with an arbitrary odd partial sum S_{2j-1}. Assume $j \leq k$. Then

$$S_{2k} - S_{2j-1} = \bigl(-a_{2j} + a_{2j+1}\bigr) + \bigl(-a_{2j+2} + a_{2j+3}\bigr) + \cdots + \bigl(-a_{2k-2} + a_{2k-1}\bigr) + \bigl(-a_{2k}\bigr).$$

In the last expression each of the numbers in parenthesis is nonpositive. Therefore, as a sum of nonpositive numbers, $S_{2k} - S_{2j-1}$ is nonpositive. That is $S_{2k} \leq S_{2j-1}$ whenever $j \leq k$.

Assume now that $j > k$. Then

$$S_{2j-1} - S_{2k} = \bigl(a_{2k+1} - a_{2k+2}\bigr) + \bigl(a_{2k+3} - a_{2k+4}\bigr) + \cdots + \bigl(a_{2j-3} - a_{2j-2}\bigr) + \bigl(a_{2j-1}\bigr).$$

In the last expression each of the numbers in parenthesis is nonnegative. Therefore $S_{2j-1} - S_{2k} \geq 0$. That is, $S_{2k} \leq S_{2j-1}$ whenever $j > k$. Thus we conclude that

$$S_{2k} \leq S_{2j-1} \quad \text{for all} \quad j, k \in \mathbb{N}. \tag{3.5}$$

In particular (3.5) means that $\{S_{2n}\}_{n=1}^{+\infty}$ is bounded above and that each S_{2j-1}, $j \in \mathbb{N}$, is its upper bound. Since the sequence $\{S_{2n}\}_{n=1}^{+\infty}$ is also non-decreasing, the Monotone Convergence Theorem, Theorem 1.16, implies that $\{S_{2n}\}_{n=1}^{+\infty}$ converges to its least upper bound, call it S. Consequently

$$S_{2k} \leq S \leq S_{2j-1} \quad \text{for all} \quad j, k \in \mathbb{N}. \tag{3.6}$$

For any pair of consecutive natural numbers $n-1, n$ one of them is even and one of them is odd. Therefore the inequalities in (3.6) imply that

$$|S_n - S| \leq |S_n - S_{n-1}| = a_n \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.7}$$

Let $\epsilon > 0$ be arbitrary. Let $n \in \mathbb{N}$ be such that $n > N(\epsilon)$. Then by (3.4) we conclude that

$$a_n < \epsilon \tag{3.8}$$

Combining the inequalities (3.7) and (3.8) we conclude that

$$|S_n - S| < \epsilon.$$

Thus we have proved that for each $\epsilon > 0$ there exists $N(\epsilon)$ such that

$$n \in \mathbb{N}, \quad n > N(\epsilon) \qquad \Rightarrow \qquad |S_n - S| < \epsilon\,.$$

This proves that the sequence $\{S_n\}_{n=1}^{+\infty}$ converges and therefore the alternating series converges. □

Example 3.14. Prove that the series in (3.1) converges. The series in (3.1) is called the *alternating harmonic series*.

Solution. We verify two conditions of the Alternating Series Test:

$$a_{n+1} \leq a_n \quad \text{since} \quad \frac{1}{n+1} < \frac{1}{n}, \quad \text{for all} \quad n \in \mathbb{N},$$

$$\lim_{n \to +\infty} \frac{1}{n} = 0 \quad \text{is easy to prove by definition.}$$

Thus the Alternating Series Test implies that the alternating harmonic series converges. □

REMARK 3.15. The Alternating Series Test does not apply to the series in (3.2) since the sequence of numbers

$$1,\ 1,\ \frac{1}{3},\ \frac{1}{2},\ \frac{1}{5},\ \frac{1}{3},\ \frac{1}{7},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{5},\ \frac{1}{9},\ \frac{1}{6},\ \ldots,\ \frac{4}{n\big(3+(-1)^{n+1}\big)},\ \ldots$$

is not non-increasing. Further exploration of the series in (3.2) would show that it diverges.

The Alternating Series Test does not apply to the series in (3.3) since this series does not satisfy the condition (ii):

$$\lim_{n\to+\infty}\frac{n+1}{n}=1\neq 0\,.$$

Again this series is divergent by the Test for Divergence.

EXERCISE 3.16. Determine whether the given series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{+\infty}\cos\left(n\pi+\frac{1}{n}\right)$ (b) $\displaystyle\sum_{n=0}^{+\infty}\sin\left(n\frac{\pi}{2}\right)$ (c) $\displaystyle\sum_{n=1}^{+\infty}\sin\left(n\pi-\frac{1}{n}\right)$

(d) $\displaystyle\sum_{n=1}^{+\infty}\frac{1}{n}\cos\left(n\pi+\frac{1}{n}\right)$ (e) $\displaystyle\sum_{n=1}^{+\infty}\ln\left(1-\frac{(-1)^n}{n}\right)$ (f) $\displaystyle\sum_{n=1}^{+\infty}\frac{1}{n}\sin\left(n\frac{\pi}{2}\right)$

(g) $\displaystyle\sum_{n=1}^{+\infty}\sin\left(n\frac{\pi}{2}+\frac{1}{n}\right)$ (h) $\displaystyle\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n-(-1)^n}$ (i) $\displaystyle\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{2n-(-1)^n}$

Several of the exercises in the next section use the Alternating Series Test for convergence. Do those exercises as well.

3.4. Absolute and Conditional Convergence. In the previous section we proved that the alternating harmonic series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots+(-1)^{n+1}\frac{1}{n}+\cdots=\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{n}\quad\text{converges.}\tag{3.9}$$

Later on we will see that the sum of this series is $\ln 2$.

Talking about infinite series in class I have often used the analogy with an infinite column in a spreadsheet and finding its sum. A series with positive and negative terms one can interpret as balancing a checkbook with (infinitely) many deposits and withdrawals. Looking at the alternating harmonic series (3.9) we see a sequence of alternating deposits and withdrawals, infinitely many of them. What we proved in the previous section tells that under two conditions on the deposits and withdrawals, although it has infinitely many transactions, this checkbook can be balanced.

Now comes the first surprising fact! Let's calculate how much has been deposited to this account:

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots+\frac{1}{2n-1}+\cdots=\sum_{n=1}^{+\infty}\frac{1}{2n-1}\,.$$

Applying the Limit Comparison Test with the harmonic series it is easy to conclude this series diverges. Looking at the withdrawals we see

$$-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\cdots-\frac{1}{2n-1}-\cdots=-\frac{1}{2}\sum_{n=1}^{+\infty}\frac{1}{n}\,.$$

Again this is a divergent series. This is certainly a suspicious situation: Dealing with an account to which an infinite amount of money has been deposited and an infinite amount of money has been withdrawn. A simpler way to look at this is to look at the total amount of money that went through this account (one can call this amount the total "activity" in the account):

$$\sum_{n=1}^{+\infty}\left|(-1)^{n+1}\frac{1}{n}\right| = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{n}+\cdots \tag{3.10}$$

This is the harmonic series which is divergent.

Since we know that an infinite amount of money has been deposited to this account we might want to get into the spending mood sooner. So, we rearrange the deposits and the withdrawals; we do two withdrawals after each deposit, keeping the amounts the same. This results in the following infinite series:

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\frac{1}{7}-\frac{1}{14}-\frac{1}{16}+\cdots. \tag{3.11}$$

In any real life checking account just rearranging the deposits and the withdrawals might result in an occasional low balance but the final balance will remain the same. Amazingly this is not always the case with infinite series! (This is the second surprising fact!) For example, the series in (3.11) and the series in (3.9) have identical terms which are arranged differently; in Example 3.14 we proved that the series (3.9) converges and next we will show that the series (3.11) also converges but to a different number.

To be specific denote the terms of the series (3.11) by $b_n, n \in \mathbb{N}$. Then

$$b_{3k-2} = \frac{1}{2k-1}, \quad b_{3k-1} = -\frac{1}{4k-2}, \quad b_{3k} = -\frac{1}{4k}, \quad k \in \mathbb{N}.$$

It is clear that the series (3.11) has the same terms as the alternating harmonic series. The terms of the alternating harmonic series have been reordered. For $k \in \mathbb{N}$, the term at the positions $2k-1$ (odd terms) in the alternating harmonic series is at the position $3k-2$ in the series (3.11), the term which is at the position $4k-2$ (a "half" of the even terms) in the alternating harmonic series is at the position $3k-1$ in the series (3.11) and the term which is at the position $4k$ (another "half" of the even terms) in the alternating harmonic series is at the position $3k$ in the series (3.11).

The following calculation indicates that the sum of the series in (3.11) is $1/2$ of the sum of the alternating harmonic series in (3.9). Let us calculate the $3n$-th partial sum of the series (3.11). Since this is a finite sum we can rearrange terms as we please. Here is the calculation

$$\begin{aligned} S_{3n} &= 1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\cdots+\frac{1}{2n-1}-\frac{1}{4n-2}-\frac{1}{4n} \\ &= \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\cdots+\frac{1}{4n-2}-\frac{1}{4n} \\ &= \frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}\right) \end{aligned}$$

Hence, $3n$-th partial sum of the series (3.11) is identical to one-half of the $2n$-th partial sum of the alternating harmonic series. Since the sum of the alternating harmonic series is $\ln 2$ we have

$$\lim_{n\to+\infty} S_{3n} = \frac{\ln 2}{2}.$$

Since

$$S_{3n+1} = S_{3n} + \frac{1}{2n+1} \quad \text{and} \quad S_{3n+2} = S_{3n} + \frac{1}{2n+1} - \frac{1}{4n+2} = S_{3n} + \frac{1}{4n+2},$$

we conclude that

$$\lim_{n\to+\infty} S_{3n+1} = \lim_{n\to+\infty} S_{3n+2} = \lim_{n\to+\infty} S_{3n} = \frac{\ln 2}{2}.$$

From the last three equalities one can prove rigorously that

$$\lim_{n\to+\infty} S_n = \frac{\ln 2}{2}.$$

This proves that the series (3.11) converges to $(\ln 2)/2$. That is just rearrangement of terms changed the sum.

This is a remarkable observation: a change of order of summation can change the sum of an infinite series. This feature is closely related to the fact that the total activity of the account expressed in (3.10) is a divergent series. This is a motivation for the following definition.

DEFINITION 3.17. A convergent series $\sum_{n=1}^{+\infty} a_n$ is called **conditionally convergent** if the series of the absolute values of its terms $\sum_{n=1}^{+\infty} |a_n|$ is divergent.

DEFINITION 3.18. A series $\sum_{n=1}^{+\infty} a_n$ is called **absolutely convergent** if the series of the absolute values of its terms $\sum_{n=1}^{+\infty} |a_n|$ is convergent.

EXAMPLE 3.19. Prove that the series

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \cdots + (-1)^{n+1}\frac{1}{n^2} + \cdots = \sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{n^2}$$

is absolutely convergent.

SOLUTION. By the definition of absolute convergence we need to determine the convergence of the series

$$\sum_{n=1}^{+\infty}\left|(-1)^{n+1}\frac{1}{n^2}\right| = \sum_{n=1}^{+\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots$$

This is a p-series with $p = 2$. Therefore this series converges. (Notice that at the beginning of Section 3.1 we proved that this series converges by comparing it to a telescoping series.) □

REMARK 3.20. One can interpreted the series in Example 3.19 as a checking account with infinitely many alternating deposits and withdrawals. In this case the total activity of the account is a convergent series. Consequently the total amount deposited

$$1 + \frac{1}{9} + \frac{1}{25} + \cdots + \frac{1}{(2n-1)^2} + \cdots = \sum_{n=1}^{+\infty}\frac{1}{(2n-1)^2} \tag{3.12}$$

and the total amount withdrawn

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{36}+\cdots+\frac{1}{(2n)^2}+\cdots=\sum_{n=1}^{+\infty}\frac{1}{(2n)^2}=\frac{1}{4}\sum_{n=1}^{+\infty}\frac{1}{n^2} \tag{3.13}$$

are both convergent series. As we can see, the total amount withdrawn is 1/4 of the total activity of the account. We mentioned before that (we can not prove it in this course)

$$\sum_{n=1}^{+\infty}\frac{1}{n^2}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\cdots=\frac{\pi^2}{6}.$$

Therefore

$$\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{n^2}=1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\cdots=\frac{3}{4}\frac{\pi^2}{6}-\frac{1}{4}\frac{\pi^2}{6}=\frac{1}{2}\frac{\pi^2}{6}=\frac{\pi^2}{12}$$

THEOREM 3.21. *If a series* $\sum_{n=1}^{+\infty} a_n$ *is absolutely convergent, then it is convergent.*

PROOF. Assume that $\sum_{n=1}^{+\infty} a_n$ is absolutely convergent, that is assume that $\sum_{n=1}^{+\infty} |a_n|$ is convergent. Then the algebra of convergent series the series $\sum_{n=1}^{+\infty} 2\,|a_n|$ is convergent. Since $-|a_n| \le a_n \le |a_n|$, we conclude that

$$0 \le a_n + |a_n| \le 2\,|a_n| \quad \text{for all} \quad n = 1, 2, 3, \ldots$$

By the Comparison Test it follows that the series $\sum_{n=1}^{+\infty}\big(a_n + |a_n|\big)$ is convergent. The algebra of convergent series implies that the series

$$\sum_{n=1}^{+\infty}\Big(\big(a_n+|a_n|\big)-|a_n|\Big)=\sum_{n=1}^{+\infty}a_n$$

is also convergent. □

The following stronger versions of the Ratio and the Root test can be applied to any series to determine whether a series converges absolutely or it diverges.

THEOREM 3.22 (The Ratio Test). *Let* $\sum_{n=1}^{+\infty} a_n$ *be a series for which* $\lim_{n\to+\infty}\frac{|a_{n+1}|}{|a_n|} = R$. *Then*

(a) *If* $R < 1$, *then the series converges absolutely.*
(b) *If* $R > 1$, *then the series diverges.*

THEOREM 3.23 (The Root Test). *Let* $\sum_{n=1}^{+\infty} a_n$ *be a series for which* $\lim_{n\to+\infty}\sqrt[n]{|a_n|} = R$. *Then*

(a) *If* $R < 1$, *then the series converges absolutely.*
(b) *If* $R > 1$, *then the series diverges.*

Notice that if the root or the ratio test apply to a series, then series either converges absolutely or diverges. This implies that if a series converges conditionally, then either

$$\lim_{n\to+\infty}\frac{|a_{n+1}|}{|a_n|}=1 \quad\text{or}\quad \lim_{n\to+\infty}\frac{|a_{n+1}|}{|a_n|} \text{ does not exist,}$$

and also

$$\lim_{n\to+\infty}\sqrt[n]{|a_n|}=1 \quad\text{or}\quad \lim_{n\to+\infty}\sqrt[n]{|a_n|} \text{ does not exist.}$$

In other words, the root and the ratio test cannot lead to a conclusion that a series converges conditionally.

It turns out that our only tool which can be used to conclude conditional convergence is the alternating series test.

Exercise 3.24. Determine whether the given series converges conditionally, converges absolutely or diverges.

(a) $\displaystyle\sum_{n=0}^{+\infty}\frac{\cos(n\pi)}{n^2+1}$ (b) $\displaystyle\sum_{n=0}^{+\infty}\frac{\sin(n\pi/2)}{n+1}$ (c) $\displaystyle\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{\sqrt{n}}$ (d) $\displaystyle\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n\sqrt{n}}$

(e) $\displaystyle\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n^p}$ (f) $\displaystyle\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{e^{1/n}}{n}$ (g) $\displaystyle\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{n^n}{n!}$ (h) $\displaystyle\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{\sqrt{n}}{n+1}$

(i) $\displaystyle\sum_{n=2}^{+\infty}\frac{(-1)^{n+1}}{\ln n}$ (j) $\displaystyle\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{\ln n}{n}$ (k) $\displaystyle\sum_{n=1}^{+\infty}(-1)^{n+1}e^{1/n}$ (l) $\displaystyle\sum_{n=1}^{+\infty}(-1)^{n+1}\ln\frac{n+1}{n}$

In problem (e) determine all the values of p for which the series converges absolutely, converges conditionally and diverges.

Exercise 3.25. Determine whether the given series converges conditionally, converges absolutely or diverges.

(a) $\displaystyle\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{(\sin n)^2}{n^2}$ (b) $\displaystyle\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{4}{2n+3+(-1)^n}$

(c) $\displaystyle\sum_{n=1}^{+\infty}(-1)^{n+1}\cos\left(\frac{1}{n}\right)$ (d) $\displaystyle\sum_{n=1}^{+\infty}(-1)^{n+1}\sin\left(\frac{1}{n}\right)$

4. Infinite Series of functions

4.1. Power Series. The most important series is the **geometric series**:

$$a + a\,r + a\,r^2 + a\,r^3 + \cdots + a\,r^n + \cdots = \sum_{n=0}^{+\infty} a\,r^n\,.$$

If $-1 < r < 1$ the geometric series converges. Moreover, we proved

$$\sum_{n=0}^{+\infty} a\,r^n = a + a\,r + a\,r^2 + a\,r^3 + \cdots + a\,r^n + \cdots = \frac{a}{1-r} \quad \text{for} \quad -1 < r < 1\,. \tag{4.1}$$

Replacing r by x and letting $a = 1$ we can rewrite the formula in (4.1) as

$$\sum_{n=0}^{+\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \frac{1}{1-x} \quad \text{for} \quad -1 < x < 1\,. \tag{4.2}$$

The formula (4.2) can be viewed as a representation of the function

$$f(x) = \frac{1}{1-x}, \quad -1 < x < 1,$$

as an infinite series of powers of x: $1 = x^0, x, x^2, x^3, \ldots$:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{+\infty} x^n \quad \text{for} \quad -1 < x < 1\,.$$

You will agree that the (non-negative) integer powers of x are very simple functions. Therefore, it is natural to explore the following question:

Q1: Which functions can be represented as infinite series of constant multiples of (non-negative) integer powers of x?

In other words: Which functions $x \mapsto f(x)$ can be represented as

$$f(x) = a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + \cdots + a_n\,x^n + \cdots = \sum_{n=0}^{+\infty} a_n\,x^n \quad \text{for} \quad ? < x < ?\,.$$

The infinite series

$$a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + \cdots + a_n\,x^n + \cdots = \sum_{n=0}^{+\infty} a_n\,x^n \tag{4.3}$$

is called a **power series**.

The first question to answer about a power series is:

Q2: For which real numbers x does the power series converge?

Since we are working with the powers of x and since there is no restriction on the signs of a_n and x, we can use Theorems 3.22 and 3.23 (the ratio and root test) to determine the absolute convergence of the power series (4.3). To apply Theorem 3.22 we calculate

$$\lim_{n\to+\infty} \frac{|a_{n+1}|\,|x|^{n+1}}{|a_n|\,|x|^n} = \lim_{n\to+\infty} \frac{|a_{n+1}|\,|x|}{|a_n|} = |x| \lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|}.$$

Assume that

$$\lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = L. \tag{4.4}$$

If $L = 0$, then Theorem 3.22 implies that the series (4.3) converges for all real numbers x. If $L > 0$, then Theorem 3.22 implies that the series (4.3)

$$\text{converges absolutely for} \quad |x|\,L < 1, \quad \text{that is for} \quad -\frac{1}{L} < x < \frac{1}{L}$$

$$\text{diverges for} \quad |x|\,L > 1, \quad \text{that is for} \quad x < -\frac{1}{L} \quad \text{or} \quad x > \frac{1}{L}$$

If the limit in (4.4) does not exist, then no conclusion about the convergence or divergence can be deduced.

To apply Theorem 3.23 we calculate

$$\lim_{n\to+\infty} \sqrt[n]{|a_n|\,|x|^n} = |x| \lim_{n\to+\infty} \sqrt[n]{|a_n|}.$$

Assume that

$$\lim_{n\to+\infty} \sqrt[n]{|a_n|} = L. \tag{4.5}$$

If $L = 0$, then Theorem 3.23 implies that the series (4.3) converges for all real numbers x. If $L > 0$, then Theorem 3.23 implies that the series (4.3)

$$\text{converges absolutely for} \quad |x|\,L < 1, \quad \text{that is for} \quad -\frac{1}{L} < x < \frac{1}{L}$$

$$\text{diverges for} \quad |x|\,L > 1, \quad \text{that is for} \quad x < -\frac{1}{L} \quad \text{or} \quad x > \frac{1}{L}$$

If the limit in (4.5) does not exist, then no conclusion about the convergence or divergence can be deduced.

Example 4.1. Consider the power series

$$\frac{1}{0!} + \frac{1}{1!}\,x + \frac{1}{2!}\,x^2 + \frac{1}{3!}\,x^3 + \cdots + \frac{1}{n!}\,x^n + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}\,x^n.$$

In this example $a_n = 1/n!$, $n = 0, 1, 2, \ldots$. We calculate

$$L = \lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to+\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to+\infty} \frac{1}{n+1} = 0.$$

Consequently the given power series converges absolutely for every $x \in \mathbb{R}$.

Example 4.2. Consider the power series

$$1 + 2x + 3\,x^2 + 4\,x^3 + \cdots + (n+1)\,x^n + \cdots = \sum_{n=0}^{\infty} (n+1)\,x^n.$$

Here $a_n = n+1$, $n = 0, 1, 2, \ldots$ and we calculate

$$L = \lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to+\infty} \frac{n+2}{n+1} = 1.$$

Consequently the given power series converges absolutely for every $x \in (-1, 1)$. Clearly the series diverges for $x = -1$ and for $x = 1$.

EXAMPLE 4.3. Consider the power series

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots + (-1)^{n+1}\frac{1}{n}x^n + \cdots = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{n}x^n.$$

Here $a_0 = 0$ and $a_n = (-1)^{n+1}1/n$, $n = 1, 2, \ldots$. We calculate

$$L = \lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to+\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to+\infty} \frac{n}{n+1} = 1.$$

Consequently the given power series converges absolutely for every $x \in (-1, 1)$. Clearly the series diverges for $x = -1$ and converges conditionally for $x = 1$.

EXAMPLE 4.4. Consider the power series

$$1 + \frac{1}{2}x + \frac{1}{2^2}x^2 + \frac{1}{2^3}x^3 + \cdots + \frac{1}{2^n}x^n + \cdots = \sum_{n=0}^{\infty}\frac{1}{2^n}x^n. \tag{4.6}$$

Here $a_n = 2^{-n}, n = 0, 1, 2, \ldots$. We calculate

$$L = \lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to+\infty} \frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}} = \lim_{n\to+\infty} \frac{1}{2} = \frac{1}{2}.$$

Consequently the given power series converges absolutely for every $x \in (-2, 2)$. Clearly the series diverges for $x = -2$ and for $x = 2$.

Notice that we can actually calculate the sum of this series using the following substitution (or you can call this a trick). Substitute $u = x/2$ in (4.6). Then (4.6) becomes

$$1 + u + u^2 + u^3 + \cdots + u^n + \cdots = \sum_{n=0}^{\infty} u^n. \tag{4.7}$$

We know that the sum of the series in (4.7) is $1/(1-u)$ for $u \in (-1, 1)$, that is,

$$1 + u + u^2 + u^3 + \cdots + u^n + \cdots = \sum_{n=0}^{\infty} u^n = \frac{1}{1-u}, \quad u \in (-1, 1).$$

Substituting back $u = x/2$ we get:

$$1 + \frac{1}{2}x + \frac{1}{2^2}x^2 + \frac{1}{2^3}x^3 + \cdots + \frac{1}{2^n}x^n + \cdots = \sum_{n=0}^{\infty}\frac{1}{2^n}x^n = \frac{2}{2-x}, \quad x \in (-2, 2).$$

EXAMPLE 4.5. Consider the power series

$$\frac{1}{1}x + \frac{1}{4}x^2 + \frac{1}{9}x^3 + \cdots + \frac{1}{n^2}x^n + \cdots = \sum_{n=1}^{\infty}\frac{1}{n^2}x^n.$$

We calculate

$$L = \lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to+\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to+\infty} \frac{n^2}{(n+1)^2} = 1.$$

Consequently the given power series converges absolutely for every $x \in (-1,1)$. For $x = 1$ we get the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Therefore, for $x = 1$ the given power series converges. For $x = -1$ we get the alternating series which converges absolutely. Therefore the given power series converges absolutely on $[-1,1]$.

The following theorem answers the question **Q2** above.

THEOREM 4.6. *Let*

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots = \sum_{n=0}^{+\infty} a_n x^n$$

be a power series. Then one of the following three cases holds.

(A) *The power series converges absolutely for all* $x \in \mathbb{R}$.
(B) *There exists* $r > 0$ *such that the power series converges absolutely for all* $x \in (-R,R)$ *and diverges for all* x *such that* $|x| > R$.
(C) *The power series diverges for all* $x \neq 0$. *For* $x = 0$ *it is trivial that the power series converges.*

The set on which a power series converges is called the *interval of convergence*. The number $R > 0$ in Theorem 4.6 (B) is called the *radius of convergence*. In the case (A) in Theorem 4.6 we write $R = +\infty$. In the case (C) in Theorem 4.6 we write $R = 0$.

REMARK 4.7. In the case (B) in Theorem 4.6 the convergence of the power series at the points $x = R$ and $x = -R$ must be determined by studying the infinite series

$$\sum_{n=0}^{+\infty} a_n R^n \quad \text{and} \quad \sum_{n=0}^{+\infty} a_n (-R)^n.$$

Examples in this section show that the interval of convergence of a power series can have any of these four forms $(-R,R)$, $(-R,R]$, $[-R,R)$ and $[-R,R]$.

4.2. Functions Represented as Power Series. The following theorem lists properties of functions defined by a power series.

THEOREM 4.8. *Let* $R > 0$ *be the radius of convergence of the power series*

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots = \sum_{n=0}^{+\infty} a_n x^n.$$

Then the function f *defined on* $(-R,R)$ *by*

$$f(x) := a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots = \sum_{n=0}^{+\infty} a_n x^n, \quad -R < x < R,$$

has the following three properties:

(a) *The function* f *is continuous on* $(-R,R)$.

(b) *The derivative $f'(x)$ exists for all $x \in (-R, R)$ and*

$$f'(x) = a_1 + 2a_2\,x + 3a_3\,x^2 + \cdots + na_n\,x^{n-1} + (n+1)a_{n+1}x^n + \cdots = \sum_{n=0}^{+\infty} (n+1)a_{n+1}\,x^n.$$

(c) *The function f has derivatives of all orders $1, 2, 3, \ldots,$ at all points of $(-R, R)$. In particular*

$$f(0) = a_0,\ f'(0) = a_1,\ f''(0) = 2\,a_2,\ f'''(0) = 3\cdot 2\,a_3,\ \ldots,\ f^{(n)}(0) = n!\,a_n,\ \ldots. \tag{4.8}$$

(d) *For all $x \in (-R, R)$ we have*

$$\int_0^x f(t)dt = a_0 x + \frac{a_1}{2}\,x^2 + \frac{a_2}{3}\,x^3 + \cdots + \frac{a_{n-1}}{n}\,x^n + \frac{a_n}{n+1}\,x^{n+1} + \cdots = \sum_{n=1}^{+\infty} \frac{a_{n-1}}{n}\,x^n.$$

THEOREM 4.9. *Let $R > 0$ be the radius of convergence of the power series*

$$a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + \cdots + a_n\,x^n + \cdots = \sum_{n=0}^{+\infty} a_n\,x^n.$$

Let f be the function defined on $(-R, R)$ by

$$f(x) := a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + \cdots + a_n\,x^n + \cdots = \sum_{n=0}^{+\infty} a_n\,x^n, \quad -R < x < R.$$

If the series

$$\sum_{n=0}^{+\infty} a_n\,R^n$$

converges, then the limit $\lim_{x\uparrow R} f(x)$ exists and

$$\lim_{x\uparrow R} f(x) = \sum_{n=0}^{+\infty} a_n\,R^n.$$

If the series

$$\sum_{n=0}^{+\infty} a_n\,(-R)^n$$

converges, then the limit $\lim_{x\downarrow R} f(x)$ exists and

$$\lim_{x\downarrow R} f(x) = \sum_{n=0}^{+\infty} a_n\,(-R)^n.$$

EXAMPLE 4.10. By (4.2) we have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \quad \text{for} \quad -1 < x < 1\,. \tag{4.9}$$

Thus the function $f(x) = 1/(1-x)$ defined for $x \in (-1, 1)$ can be represented by a power series. Applying Theorem 4.8 we get

$$\frac{1}{(1-x)^2} = 1 + 2\,x + 3\,x^2 + 4\,x^3 + \cdots + n\,x^{n-1} + (n+1)\,x^n + \cdots \quad \text{for} \quad -1 < x < 1\,.$$

EXAMPLE 4.11. Substituting $-x$ for x in (4.9) we get

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots \quad \text{for} \quad -1 < x < 1\,. \tag{4.10}$$

Thus the function $f(x) = 1/(1+x)$ defined for $x \in (-1,1)$ can be represented by a power series. Applying Theorem 4.8 (d) we get

$$\ln(1+x) = \int_0^x \frac{1}{1+t}dt = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots + (-1)^{n+1}\frac{1}{n}x^n + \cdots \quad \text{for} \quad -1 < x < 1\,.$$

For $x = 1$ the above series is the alternating harmonic series which converges conditionally. By Theorem 4.9 we have

$$\lim_{x\uparrow 1} \ln(1+x) = \sum_{n=0}^{+\infty} (-1)^{n+1}\frac{1}{n}.$$

Since the function $\ln(1+x)$ is continuous at $x = 1$ we have

$$\lim_{x\uparrow 1} \ln(1+x) = \ln 2.$$

Thus we found the sum of the alternating harmonic series

$$\sum_{n=0}^{+\infty} (-1)^{n+1}\frac{1}{n} = \ln 2.$$

EXAMPLE 4.12. Substituting x^2 for x in (4.10) we get

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots \quad \text{for} \quad -1 < x < 1\,.$$

Thus the function $f(x) = 1/(1+x^2)$ defined for $x \in (-1,1)$ can be represented by a power series. Applying Theorem 4.8 (d) we get

$$\arctan(x) = \int_0^x \frac{1}{1+t^2}dt = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots + (-1)^{n+1}\frac{1}{2n-1}x^{2n-1} + \cdots \quad \text{for} \quad -1 < x < 1\,.$$

with $x = 1$ the above series is

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^{n+1}\frac{1}{2n-1} + \cdots$$

is a conditionally convergent alternating series. By Theorem 4.9 we have

$$\lim_{x\uparrow 1} \arctan x = \sum_{n=1}^{+\infty} (-1)^{n-1}\frac{1}{2n-1}.$$

We did not prove it, but it can be proved that $\arctan x$ is a continuous function. Therefore

$$\lim_{x\uparrow 1} \arctan x = \arctan 1 = \frac{\pi}{2}.$$

Thus we found a representation of π as an infinite sum

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^{n+1}\frac{1}{2n-1} + \cdots = \sum_{n=1}^{+\infty} (-1)^{n-1}\frac{1}{2n-1}.$$

4.3. Taylor series at 0 (Maclaurin series). In the preceding section we found power series representations for several well known functions. It turns out that all well known functions can be represented as power series. The key step in finding the power series representation of elementary functions are formulas (4.8) which establish the relationship between the coefficients a_n, $n = 0, 1, 2, \dots$, of a power series and the derivatives of the function f which is represented by that power series. We rewrite formulas (4.8) as

$$a_0 = f(0), \quad a_1 = f'(0), \quad a_2 = \frac{1}{2!}f''(0), \quad a_3 = \frac{1}{3!}f^{(3)}(0), \ \dots, \ a_n = \frac{1}{n!}f^{(n)}(0), \dots . \tag{4.11}$$

Let $a > 0$ and let f be a function defined on $(-a, a)$. Assume that f has all derivatives on $(-a, a)$. Then the series power series

$$f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f^{(3)}(0)x^3 + \cdots + \frac{1}{n!}f^{(n)}(0)x^n + \cdots = \sum_{n=0}^{+infty} \frac{1}{n!}f^{(n)}(0)x^n$$

is called *Taylor series at* 0 or *Maclaurin series* of f.

Using formulas (4.11) it is not difficult to calculate a Maclaurin series for a given function. The difficulties arise in proving that the function defined by such power series is identical to the given function. Fortunately this is true for all well known functions.

EXAMPLE 4.13. Let $f(x) = e^x = \exp(x)$, $x \in \mathbb{R}$. Then $f^{(n)}(x) = e^x$ for all $n = 0, 1, 2, \dots$. Therefore the coefficients of the Maclaurin series for the function exp are $a_n = 1/n!$ and it can be proved that for all $x \in \mathbb{R}$ we have

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots .$$

EXAMPLE 4.14. Let $f(x) = \sin(x)$, $x \in \mathbb{R}$. Then

$$f'(x) = \cos(x), \quad f''(x) = -\sin(x), \quad f^{(3)}(x) = -\cos(x), \quad f^{(4)}(x) = \sin(x).$$

Consequently,

$$f^{(2k)}(0) = 0, \quad f^{(2k+1)}(0) = (-1)^k, \quad k = 0, 1, 2, \dots .$$

Therefore the coefficients of the Maclaurin series for the function sin are

$$a_{2k} = 0, \quad a_{2k+1} = (-1)^k \frac{1}{(2k+1)!}, \quad k = 0, 1, 2, \dots .$$

It can be proved that for all $x \in \mathbb{R}$ we have

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots + (-1)^k \frac{1}{(2k+1)!}x^{2k+1} + \cdots .$$

EXAMPLE 4.15. Let $f(x) = \cos(x)$, $x \in \mathbb{R}$. Then

$$f'(x) = -\sin(x), \quad f''(x) = -\cos(x), \quad f^{(3)}(x) = \sin(x), \quad f^{(4)}(x) = \cos(x).$$

Consequently,

$$f^{(2k)}(0) = (-1)^k, \quad f^{(2k+1)}(0) = 0, \quad k = 0, 1, 2, \dots .$$

Therefore the coefficients of the Maclaurin series for the function cos are

$$a_{2k} = (-1)^k \frac{1}{(2k)!}, \quad a_{2k+1} = 0, \quad k = 0, 1, 2, \dots .$$

It can be proved that for all $x \in \mathbb{R}$ we have

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + (-1)^k \frac{1}{(2k)!}x^{2k} + \cdots .$$

EXAMPLE 4.16 (The Binomial Series). Let $\alpha \in \mathbb{R}$. Let $f(x) = (1+x)^\alpha$, $x \in (-1,1)$. Then

$$\begin{aligned} f'(x) &= \alpha(1+x)^{\alpha-1}, \\ f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2}, \\ f^{(3)}(x) &= \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}, \\ &\vdots \\ f^{(n)}(x) &= \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n} \\ &\vdots \end{aligned}$$

Therefore the coefficients of the Maclaurin series for the function f are

$$a_0 = 1, \quad a_n = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}, \quad n \in \mathbb{N}.$$

It can be proved that for all $x \in (-1,1)$ we have

$$(1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + \cdots .$$

This series is called *binomial series*. The reason for this name is that for $\alpha \in \mathbb{N}$ the binomial series becomes a polynomial:

$$\begin{aligned} (1+x)^1 &= 1 + x \\ (1+x)^2 &= 1 + 2x + x^2 \\ (1+x)^3 &= 1 + 3x + 3x^2 + x^3 \\ (1+x)^4 &= 1 + 4x + 6x^2 + 4x^3 + x^4 \\ (1+x)^5 &= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 \\ (1+x)^6 &= 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6 \\ &\vdots \end{aligned}$$

$$(1+x)^m = \sum_{k=0}^{m} \binom{m}{k} x^k, \quad \text{were} \quad m \in \mathbb{N} \quad \text{and} \quad \binom{m}{k} = \frac{m!}{k!(m-k)!}$$

The last formula is called the *binomial theorem*. The coefficients

$$\binom{m}{k} = \frac{m!}{k!(m-k)!} = \frac{m(m-1)\cdots(m-k+1)}{k!} \quad \text{with} \quad m,k \in \mathbb{N}, \quad 0 \le k \le m,$$

are called *binomial coefficients*. With a general $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$ the coefficients

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

are called *generalized binomial coefficients*. By definition $\binom{\alpha}{0} = 1$. With this notation the binomial series can be written as

$$(1+x)^\alpha = \sum_{k=0}^{+\infty} \binom{\alpha}{k} x^k \quad \text{for} \quad x \in (-1,1). \tag{4.12}$$

Notice that formula (4.9) is a special case of (4.12), since

$$\binom{-1}{k} = \frac{(-1)(-2)\cdots(-1-k+1)}{k!} = \frac{(-1)^k k!}{k!} = (-1)^k.$$

Notice also that differentiating (4.9) leads to

$$(1+x)^{-2} = 1 + \sum_{k=1}^{+\infty} (-1)^k (k+1)\, x^k \quad \text{for} \quad -1 < x < 1\,.$$

This is a binomial series with $\alpha = -2$. To verify this we calculate

$$\binom{-2}{k} = \frac{(-2)(-3)\cdots(-2-k+1)}{k!} = \frac{(-1)^k (k+1)!}{k!} = (-1)^k (k+1).$$

For $\alpha = 1/2$ the expression

$$\begin{aligned}\binom{1/2}{k} &= \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(\frac{1}{2}-k+1\right)}{k!} \\ &= \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2k-3}{2}\right)}{k!} \\ &= \frac{(-1)^{k-1} 1\cdot 3\cdot \cdots (2k-3)}{2^k\, k!}\end{aligned}$$

Thus

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{2^2 2!}x^2 + \frac{1\cdot 3}{2^3 3!}x^3 - \frac{1\cdot 3\cdot 5}{2^4 4!}x^4 + \frac{1\cdot 3\cdot 5\cdot 7}{2^5 5!}x^5 + \cdots \quad \text{for} \quad -1 < x < 1\,.$$

EXAMPLE 4.17. Let $f(x) = \arcsin(x),\ x \in [-1,1]$. To calculate the Maclaurin series for arcsin we notice that

$$\frac{d}{dx}\big(\arcsin(x)\big) = \frac{1}{\sqrt{1-x^2}}, \qquad x \in (-1,1).$$

Now calculate the Maclaurin series for the last function using the binomial series with $\alpha = -1/2$. For $\alpha = -1/2$ and $k \in \mathbb{N}$, we calculate

$$\begin{aligned}\binom{-1/2}{k} &= \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-k+1\right)}{k!} \\ &= \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2k-1}{2}\right)}{k!} \\ &= (-1)^k \frac{1\cdot 3\cdot \cdots \cdot (2k-1)}{2^k\, k!}\end{aligned}$$

Thus

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{1\cdot 3}{2^2 2!}x^2 + \frac{1\cdot 3\cdot 5}{2^3 3!}x^3 - \frac{1\cdot 3\cdot 5\cdot 7}{2^4 4!}x^4 + \cdots \quad \text{for} \quad -1 < x < 1\,,$$

that is,

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{k=1}^{+\infty} (-1)^k \frac{1\cdot 3\cdot \cdots \cdot (2k-1)}{2^k\, k!}\ x^k,$$

or using the notation of double factorials

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{k=1}^{+\infty} (-1)^k \frac{(2k-1)!!}{(2k)!!}\, x^k.$$

Substituting $-x^2$ instead of x in the above formula we get

$$\frac{1}{\sqrt{1-x^2}} = 1 + \sum_{k=1}^{+\infty} \frac{(2k-1)!!}{(2k)!!}\, x^{2k}, \quad \text{for} \quad -1 < x < 1.$$

Since

$$\int_0^x \frac{1}{\sqrt{1-t^2}} dt = \arcsin(x),$$

integrating the last power series we get

$$\arcsin(x) = x + \sum_{k=1}^{+\infty} \frac{(2k-1)!!}{(2k+1)(2k)!!}\, x^{2k+1} = \sum_{k=0}^{+\infty} \frac{\binom{2k}{k}}{4^k(2k+1)}\, x^{2k+1}, \quad \text{for} \quad -1 < x < 1$$

It is interesting to note that the above expansion holds at both endpoints $x = -1$ and $x = 1$. To prove this we need to recall Theorem 4.8 (a) and prove that the series

$$1 + \sum_{k=1}^{+\infty} \frac{(2k-1)!!}{(2k+1)(2k)!!}$$

converges. This series converges by The Comparison Test. (**Hint:** Prove by mathematical induction that $\dfrac{(2k-1)!!}{(2k)!!} < \dfrac{1}{\sqrt[3]{k}}$ for all $k \in \mathbb{N}$.) As a consequence we obtain that

$$1 + \sum_{k=1}^{+\infty} \frac{(2k-1)!!}{(2k+1)(2k)!!} = \sum_{k=0}^{+\infty} \frac{\binom{2k}{k}}{4^k(2k+1)} = \frac{\pi}{2}.$$