Limit at
infinity
examples

Limit of $f(x)$ as $x \to +\infty$

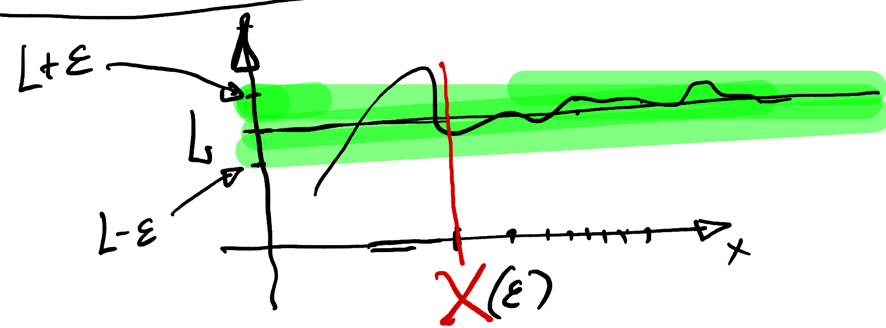
Definition Let f be a function. We say that L is a limit of f as $x \to +\infty$ if the foll. 2 cond. are satisfied:

(I) $\exists X_0 \in \mathbb{R}$ s.t $f(x)$ is defined $\forall x \geq X_0$

(II) $\forall \varepsilon > 0 \; \exists X(\varepsilon) \geq X_0$ s.t.

$$x > X(\varepsilon) \Rightarrow |f(x) - L| < \varepsilon$$

This means that $[X_0, +\infty)$ is included in the domain of f



Ex 1 $f(x) = \frac{x}{x+\sin x}$. Does this function satisfy (I)?

(I) ? $X_0 \in \mathbb{R}$ s.t. $f(x)$ defined $\forall x \geq X_0$?

$X_0 = 2$ Clearly $\frac{x}{x+\sin x}$ defined $\forall x \geq 2$

(II) Let $\varepsilon > 0$ be arbitrary

$\boxed{x+\sin x \geq 1,\ x \geq 2}$

I need $X(\varepsilon) \geq 2$ s.t.

$$\underbrace{\left|\frac{x}{x+\sin x} - 1\right|}_{\text{simplify this expression!}} < \varepsilon$$

$\boxed{\text{Have in mind that } x \geq 2}$

$$\left|\frac{x}{x+\sin x} - 1\right| \underset{alg}{=} \left|\frac{x - x - \sin x}{x+\sin x}\right| \underset{alg}{=} \left|\frac{-\sin x}{x+\sin x}\right| =$$

rules for abs

$$\left(\left|\frac{a}{b}\right| = \frac{|a|}{|b|}\right) = \frac{|-\sin x|}{x+\sin x}$$

Now we need to solve $\boxed{\frac{|-\sin x|}{x+\sin x} < \varepsilon}$ goal

This is still not simple enough to solve.

$$\underbrace{\boxed{\frac{|-\sin x|}{x+\sin x} \leq \frac{1}{x-1}}}_{\text{PIZZA-PARTY}}$$

comes to rescue! ☺

Now we just solve $\frac{1}{x-1} < \varepsilon$

New goal is

EASY

SOLVE FOR x

Summary:

S1 $\left|\frac{x}{x+\sin x}-1\right| = \frac{|1-\sin x|}{x+\sin x}$

S2 $\frac{|1-\sin x|}{x+\sin x} \leq \frac{1}{x-1}$

S3 Solve for $x \geq 2$ $\frac{1}{x-1} < \varepsilon$

$x \geq 2 \Rightarrow x-1 \geq 0$, ineq. respects reciprocal

yields: $x-1 > \frac{1}{\varepsilon}$

so $\boxed{x > \frac{1}{\varepsilon}+1}$

S4 $x \geq \underbrace{\max\left\{2, \frac{1}{\varepsilon}+1\right\}}_{X(\varepsilon)=} \Rightarrow \frac{1}{x-1} < \varepsilon$

Given $\varepsilon > 0$ arb. choose $X(\varepsilon) = \max\{2, \frac{1}{\varepsilon} + 1\}$

then $x > X(\varepsilon) \Rightarrow \left| \frac{x}{x+\sin x} - 1 \right| < \varepsilon$?

I can prove this!

Here is a proof. Assume $x > \max\{2, \frac{1}{\varepsilon} + 1\}$.

then $x \geq 2$ and $\underbrace{x > \frac{1}{\varepsilon} + 1}$.

then $x \geq 2$ and $\frac{1}{x-1} < \varepsilon$. (alg.)

G1

By (S2) and (S1)

$$\left|\frac{x}{x+\sin x}-1\right| \le \frac{1}{x-1} \quad \text{(G2)}$$

By transitivity and (G1) and (G2) I conclude

$$\left|\frac{x}{x+\sin x}-1\right| < \varepsilon$$

this completes the proof. QED!