$$\lim_{x \to +\infty} \tanh(x) = 1$$

Prove it !

Definition Let $L \in \mathbb{R}$, $D \subseteq \mathbb{R}$, $f: D \to \mathbb{R}$.

L is a limit of f as x approaches $+\infty$ if the following two conditions are satisfied:

(I) $\exists X_0 \in \mathbb{R}$ such that $[X_0, +\infty) \subseteq D$

(II) $\forall \varepsilon > 0 \; \exists X(\varepsilon) \geq X_0$ such that

$$\boxed{x > X(\varepsilon)} \Rightarrow \boxed{|f(x) - L| < \varepsilon}$$

In our example $\boxed{|th(x) - 1| < \varepsilon}$

<u>Example</u> $\boxed{\lim_{x\to+\infty} th(x) = 1}$ Recall $\boxed{th(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}}$ $\forall x \in \mathbb{R}$

Thus $\operatorname{dom} th = \mathbb{R}$. $\to$ 1 might be a better choice

(I) So, we can take $x_0 = 0$

(II) is harder. Let $\varepsilon > 0$ be arbitrary.

To find $X(\varepsilon)$ we need to simplify for $x>0$

$$|th(x) - 1| = \left|\frac{e^x - e^{-x}}{e^x + e^{-x}} - 1\right| = \left|\frac{e^x - e^{-x} - e^x - e^{-x}}{e^x + e^{-x}}\right| =$$

$$= \left|\frac{-2e^{-x}}{e^x + e^{-x}}\right| \underset{\substack{e^x + e^{-x} > 0 \\ e^{-x} > 0}}{\overset{\text{prop of abs}}{=}} \frac{2e^{-x}}{e^x + e^{-x}} \underset{\text{algebra}}{=} \frac{2}{e^{2x} + 1}$$

Thus $\boxed{|\tanh(x) - 1| = \dfrac{2}{e^{2x}+1}}$ True for $\forall x \in \mathbb{R}$ But we need only $x \geq 0$.

Now we need to solve for $x \geq 0$ $\boxed{\dfrac{2}{e^{2x}+1} < \varepsilon.}$

still too complicated use Pizza-Party to simplify

$$\frac{2}{e^{2x}+1} < \frac{2}{e^{2x}}$$

Now solve $\boxed{\dfrac{2}{e^{2x}} < \varepsilon}$ easier to solve

this is our $b(x)$ from pdf!

the learning is recognition of RESPECT among MATH obj.

true for all $x \in \mathbb{R}$

Solve for x

$$\frac{2}{e^{2x}} < \varepsilon \overset{\text{respect}}{\iff} \frac{e^{2x}}{2} > \frac{1}{\varepsilon} \iff e^{2x} > \frac{2}{\varepsilon}$$

$\varepsilon > 0$

$\ln \uparrow$ fun

$1/2 > 0$

$$\underbrace{e^{2x}}_{>0} > \underbrace{\frac{2}{\varepsilon}}_{>0} \iff 2x > \ln\left(\frac{2}{\varepsilon}\right) \iff x > \frac{1}{2}\ln\left(\frac{2}{\varepsilon}\right)$$

<u>Summarize</u>

G1 $\boxed{\left|f_n(x) - 1\right| < \frac{2}{e^{2x}} \quad \forall x \in \mathbb{R}}$

G2 $\boxed{\frac{2}{e^{2x}} < \varepsilon \iff x > \frac{1}{2}\ln\left(\frac{2}{\varepsilon}\right)}$

Set $\cancel{X}(\varepsilon) = \max\left\{\frac{1}{2}\ln\left(\frac{2}{\varepsilon}\right), 0\right\}$

all green

Now prove: $\boxed{x > \max\left\{\frac{1}{2}\ln\left(\frac{2}{\varepsilon}\right), 0\right\}} \Rightarrow \boxed{\left|\operatorname{th}(x) - 1\right| < \varepsilon}$

assume — a prove

Assume $x > \max\left\{\frac{1}{2}\ln\left(\frac{2}{\varepsilon}\right), 0\right\} \Rightarrow x > \frac{1}{2}\ln\left(\frac{2}{\varepsilon}\right) \Rightarrow \boxed{\frac{2}{e^{2x}} < \varepsilon}$

green — green — G2 — green

By G1 $\boxed{\left|\operatorname{th}(x) - 1\right| < \frac{2}{e^{2x}}}$

green

By transitivity the last two green boxes give

order

$\boxed{\left|\operatorname{th}(x) - 1\right| < \varepsilon}$

red is greenified

$$ax^2+bx+c=0$$

$\boxed{a \neq 0}$

Do you know the algebra that will separate red and green?

solving an equation is separating red and green

$$x = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

or

$$x = \frac{-b+\sqrt{b^2-4ac}}{2a}$$

All green

This is just to emphasize that red-green interplay is present throughout math