An example of limit at a

$$\lim_{x \to 2} x^3 = 8$$

__Def.__ $f: \overset{\text{domain}}{D} \to \overset{\text{codomain}}{\mathbb{R}}$, $a \in \mathbb{R}$, $L \in \mathbb{R}$

$$\lim_{x \to a} f(x) = L$$

(I) $\exists\, \delta_0 > 0$ s.t. $(a-\delta_0, a) \cup (a, a+\delta_0) \subseteq D$

(II) $\forall\, \varepsilon > 0 \quad \exists\, \delta(\varepsilon) > 0$ s.t. $\delta(\varepsilon) \le \delta_0$ and

$$\underbrace{0 < |x-a|}_{\text{excludes } a} < \delta(\varepsilon) \Rightarrow |f(x) - L| < \varepsilon$$

Example $\lim_{x \to 2} x^3 = 8$

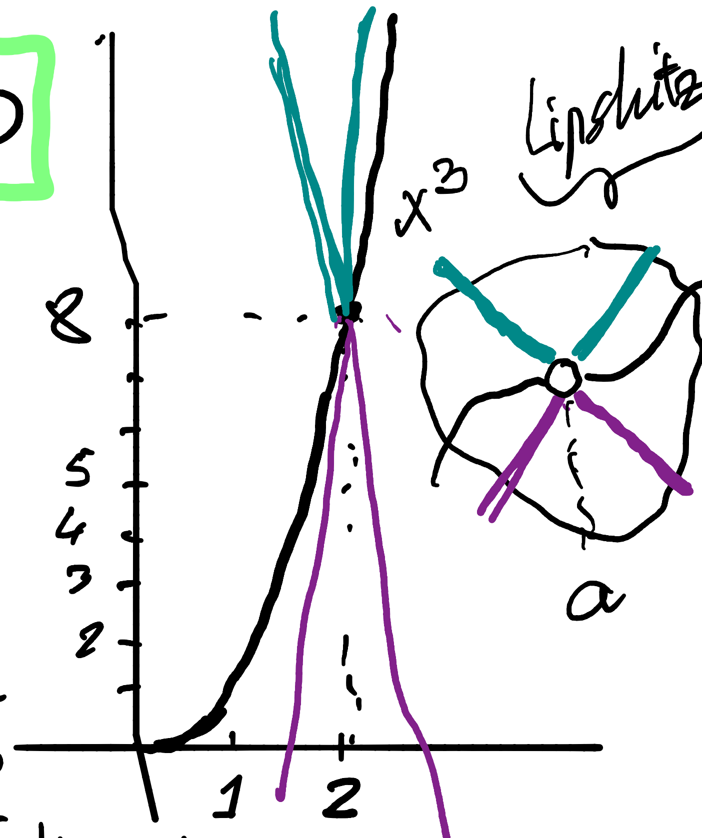
Proof. (I) $\boxed{\delta_0 = 1 > 0}$

(II) Let $\varepsilon > 0$ be arbitrary

Find $\delta(\varepsilon) \le 1$

s.t.

$$0 < |x-2| < \delta(\varepsilon) \Rightarrow \underbrace{|x^3 - 8| < \varepsilon}_{\text{solve for } |x-2|}$$



Solve $\boxed{|x^3-8|<\varepsilon}$ for $|x-2|$?

(Have in mind $\delta_0 = 1$ so, we are interested only in $x \in (2-1,2) \cup (2,2+1)$

$= (1,2) \cup (2,3)$

honestly, here, $\boxed{\begin{array}{c} x \in (1,3) \\ |x-2| < 1 \end{array}}$)

Waldo is $|x-2|$

Where is Waldo ? $\to |x-2|$

$x^3 - y^3 = (x-y)(x^2+xy+y^2) = x^3 + \cancel{x^2y} + \cancel{xy^2} - \cancel{x^2y} - \cancel{xy^2} - y^3$

$x^2 - y^2 = (x-y)(x+y)$

$$x^3 - 8 = (x-2)(x^2+2x+4)$$

please check

here is Waldo!

solution depends on green stuff

$$|x^3-8| = |x-2|\,|x^2+2x+4| \quad \forall x \in \mathbb{R}$$

only!

Solve:

$|x-2|\,|x^2+2x+4| < \varepsilon$ for $|x-2|$

only

find a solution for $|x-2|$ depending on ε only!

Pizza-Party comes to rescue!

Pizza, Party = 1

Recall $x \in (1,3)$ We consider only these x!

What is the largest value of $|x^2+2x+4|$ on the $(1,3)$.

A quick way to do this is to use the triangle inequality

$$|x^2+2x+4| \le |x|^2+2|x|+4 \le \underline{\underline{19}}.$$

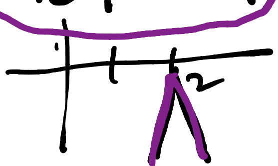
We discovered! (B/N)

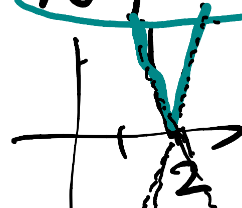
(Lipschitz) $\boxed{|x^3-8| \le 19\,|x-2| \quad \text{for all } x\in(1,3)}$

We can picture this inequality; go back to graph

$$-19\,|x-2| \le x^3-8 \le 19\,|x-2|$$

$$|y|\le b \iff -b\le y\le b \qquad b\ge 0$$





$$-19|x-2|+8 \le x^3 \le 19|x-2|+8$$

Go back to the graph of x^3.

Summarize: Need to solve $|x^3-8|<\varepsilon$ for $|x-2|$ knowing $x\in(1,3)$

Know $\boxed{|x^3-8| \le 19|x-2| \text{ for } x\in(1,3)}$

Solve $19|x-2|<\varepsilon$ for $|x-2|$

The solution is easy $|x-2|<\dfrac{\varepsilon}{19}$.

Now we can state our $\delta(\varepsilon)$:

$$\delta(\varepsilon) = \min\left\{\frac{\varepsilon}{19}, 1\right\}$$

Red = green

smells like a solution

Need to prove that $0 < |x-2| < \min\left\{\frac{\varepsilon}{19}, 1\right\}$

$\Rightarrow |x^3-8| < \varepsilon$

Green is B / W $\quad |x^3-8| \leq 19|x-2|$ for $x \in (1,3)$

Now I can prove $\Leftarrow$ using $\min\left\{\frac{\varepsilon}{19}, 1\right\}$ BIN

Proof: Assume $\boxed{0 < |x-2| < \min\left\{\frac{\varepsilon}{19}, 1\right\}}$

then $\boxed{|x-2| < \frac{\varepsilon}{19}}$ and $\boxed{|x-2| < 1}$

then $\boxed{19|x-2| < \varepsilon}$ and $\boxed{x \in (1,3)}$

By BIN G1 G2 $\boxed{|x^3-8| \leq 19|x-2|}$

by transitivity of $\leq$ G1 & G2

I conclude $\boxed{|x^3-8| < \varepsilon}$

red has been greenified