Boundedness, Monotonicity, Convergence of Sequences

May 15, 2020

All our theorems so far assumed that some sequences are convergent. For example, the

Squeeze Theorem: $a : \mathbb{N} \to \mathbb{R}$

$b : \mathbb{N} \to \mathbb{R}$

$s : \mathbb{N} \to \mathbb{R}$

Assume $\boxed{\lim_{n\to+\infty} a_n = L}$, $\boxed{\lim_{n\to+\infty} b_n = L}$

$\exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$ $n \geq n_0 \Rightarrow a_n \leq s_n \leq b_n$

Then $\boxed{\lim_{n\to+\infty} s_n = L}$

Deficiency of this theorem is that we need to know a lot of stuff

We need a theorem that will claim convergence of a sequence with fewer and simpler ASSUMPTIONS.

that would be a more powerful TOOL.

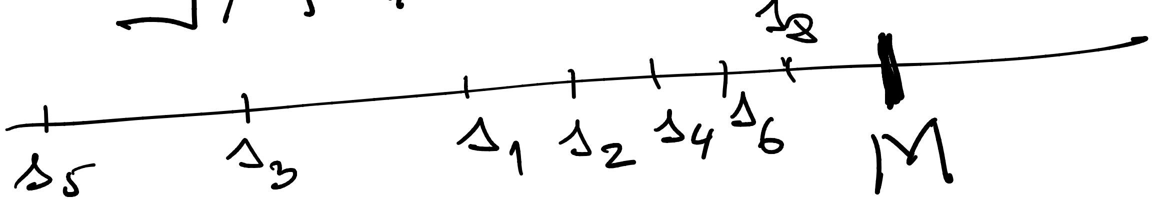
We need new concepts:

BOUNDEDNESS and MONOTONICITY

Def. A sequence $s:\mathbb{N}\to\mathbb{R}$ is called BOUNDED ABOVE

if $\exists M\in\mathbb{R}$ s.t. $\forall n\in\mathbb{N}$ $s_n \le M$

upper bound for s



Def. A sequence $s : \mathbb{N} \to \mathbb{R}$ is called BOUNDED BELOW

if $\exists m \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$ $m \leq s_n$ lower bound for s

Theorem If a sequence $\boxed{s : \mathbb{N} \to \mathbb{R}}$ $\boxed{\text{converges}}$, then it is $\boxed{\text{bounded}}$.

means bounded above and bounded below

Proof. We need to prove $\exists m, M \in \mathbb{R}$ such that

$\forall n \in \mathbb{N}$ $m \leq s_n \leq M$.

Where is our GREEN stuff? $\boxed{s \text{ converges}}$ English Green

Please, give me GREEN is Mathlish

Mathlish Green is powerfull | English green is simple

$$\boxed{\exists L \in \mathbb{R} \text{ s.t. } \forall \varepsilon > 0 \ \exists N(\varepsilon) \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N} \ n > N(\varepsilon) \Rightarrow |s_n - L| < \varepsilon}$$

Set $\varepsilon = 1$

and read Mathish GREEN

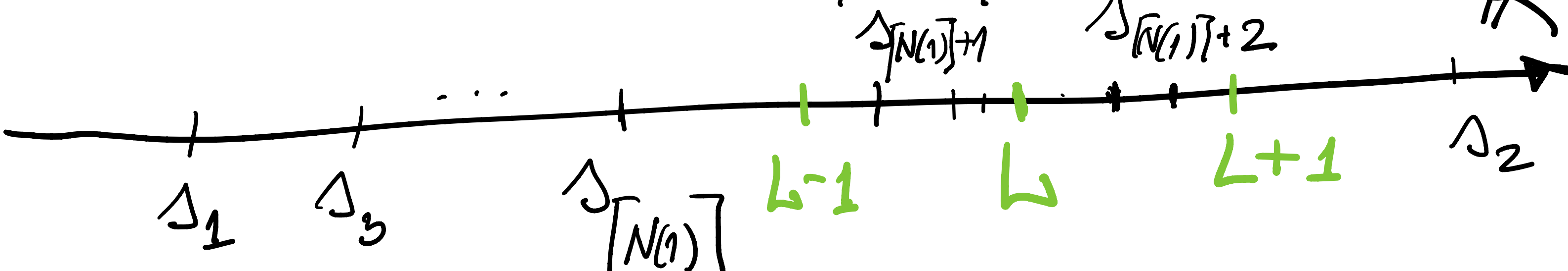
$-\varepsilon + L < \Delta_n < L + \varepsilon$

please recall BBB principle

$\exists L \in \mathbb{R}.$ $\exists N(1) \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$ $\underline{n > N(1)} \Rightarrow \underline{-1 + L < \Delta_n < L + 1}$

knowledge

There is no information in this GREEN BOX about $\Delta_1, \Delta_2, \Delta_3, \dots, \Delta_{[N(1)]}$



$$m = \min\{\overbrace{s_1, s_2, \dots, s_{\lceil N(1)\rceil}}^{\text{finitely many reals}}, L-1\}$$

$$M = \max\{\overbrace{s_1, s_2, \dots, s_{\lceil N(1)\rceil}}^{\text{finitely many reals}}, L+1\}$$

Now I can prove that

$$\forall n \in \mathbb{N} \quad m \le s_n \le M.$$

<u>Case 1</u> $n \in \{1, 2, \dots, \lceil N(1)\rceil\}$. Then by the def. of minimum we have $\boxed{m \le s_n}$ for all $n \in \{1, 2, \dots, \lceil N(1)\rceil\}$

By the def. of maximum:

$$\boxed{s_n \leq M}$$

Case 2 $n \in \mathbb{N}$ s.t. $n > \lceil N(1) \rceil$

Then by Mathish GREEN

we have $\boxed{L-1 < s_n < L+1}$

But by the def of m and M

$\boxed{m \leq L-1}$ and $\boxed{L+1 \leq M}$

Hence, by transitivity of ORDER

$$\boxed{m < s_n < M}$$

By Case 1 and Case 2:

$$\boxed{\forall n \in \mathbb{N} \quad m \le s_n \le M}$$

QED

Converse is NOT true.

$s_n = (-1)^n \quad \forall n \in \mathbb{N}$, $-1, 1, -1, 1, -1, 1, \ldots$ bounded but does not converge.

For a meal we want a convergent seq. Just bounded seq is not a full meal.

It turns out that monotonicity is the key here.

$s : \mathbb{N} \to \mathbb{R}$

<u>Def</u>: A sequence is <u>non-decreasing</u> if

$$\forall n \in \mathbb{N} \quad s_n \leq s_{n+1}$$

A sequence is <u>non-increasing</u> if

$$\forall n \in \mathbb{N} \quad s_n \geq s_{n+1}$$

A ~~one~~ name for either one of these is a MONOTONIC sequence.

Theorem

Monotone Convergence Theorem (MCT)

$\boxed{\text{A bounded, monotonic sequence converges.}}$

The proof is an amazing affair!

It uses the COMPLETENESS AXIOM of $\mathbb{R}$

the most important property of $\mathbb{R}$.