Telescopinc Series

The Divergence Test

$$\boxed{\boxed{\sum_{n=1}^{\infty} a_n \text{ CONVERGES}} \Rightarrow \boxed{\lim_{n\to\infty} a_n = 0}}$$

The converse is NOT TRUE

$$\boxed{\lim_{n\to\infty} \frac{1}{n} = 0 \text{ AND } \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}}$$

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We are studying Infinite Series

Given a sequence $a: \mathbb{N} \to \mathbb{R}$

with terms $a_1, a_2, a_3, \dots, a_n, \dots$

$$\sum_{n=1}^{\infty} a_n$$

this is how we denote the corresponding infinite series.

In fact we are asking:

Does it converge or diverge?

That is decided by looking at the sequence of its partial sums

$$\forall n \in \mathbb{N} \qquad S_n = \sum_{k=1}^{n} a_k$$

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$S_4 = a_1 + a_2 + a_3 + a_4$

$\dots$

Yesterday we PROVED that the Harmonic series diverges: Why?

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n}}$$

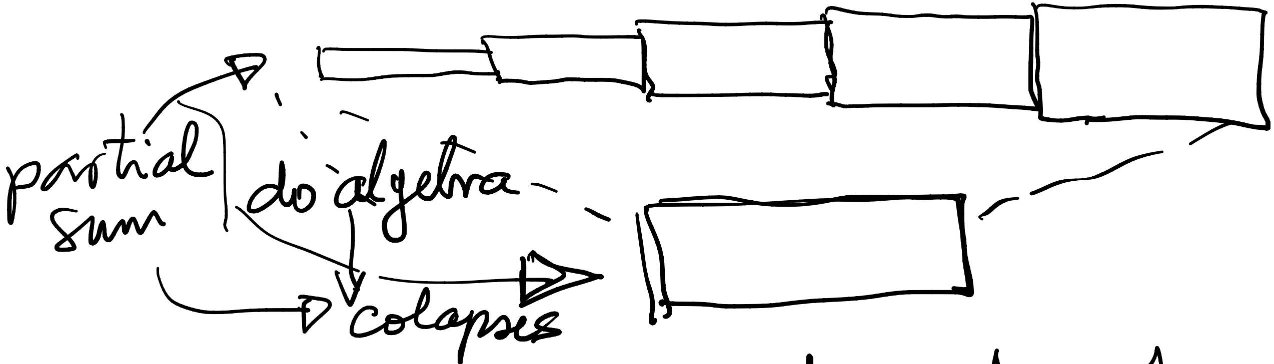
Since its partial sums so called Harmonic numbers,

the $\mathbb{N}$ $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$

is an increasing, but unbounded sequence.

Today we will learn about a usefull "series trick":

TELESCOPIC SERIES



$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} + \cdots$$

?

CONVERGES or DIVERGES ?

$$\boxed{S_n = \frac{1}{1\cdot 2} + \cdots + \frac{1}{k(k+1)} + \cdots + \frac{1}{n(n+1)}}$$

see below this sum collapses to just two terms

in green

Trick "partial fractions"

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$S_n = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \cdots + \left(\cancel{\frac{1}{k-1}} - \cancel{\frac{1}{k}}\right) + \left(\cancel{\frac{1}{k}} - \cancel{\frac{1}{k+1}}\right) + \cdots$$

$$\cdots + \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

(telescoping)

The magic is

$$\boxed{S_n = \frac{1}{1} - \frac{1}{n+1} \quad \forall n \in \mathbb{N}}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to+\infty}\left(1 - \frac{1}{n+1}\right) = 1$$

Thus we write $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$

$$\sum_{n=2}^{+\infty} \frac{1}{n^2-1} = \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \cdots + \frac{1}{n^2-1} + \cdots$$

$k>1$

$$\frac{1}{k^2-1} = \frac{1}{(k+1)(k-1)} = \boxed{\frac{1/2}{k-1} - \frac{1/2}{k+1}}$$

$$\frac{k+1-(k-1)}{(k-1)(k+1)} = \frac{2}{k^2-1}$$

$$S_n = \sum_{k=2}^{n} \frac{1}{k^2-1} = \frac{1}{2}\Bigg(\underbrace{\left(\boxed{\frac{1}{1}} - \cancel{\frac{1}{3}}\right)}_{k=2} + \underbrace{\left(\boxed{\frac{1}{2}} - \cancel{\frac{1}{4}}\right)}_{k=3} + \underbrace{\left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right)}_{k=4} +$$

$n \geq 2$

$$+ \underbrace{\left(\cancel{\frac{1}{4}} - \frac{1}{6}\right)}_{k=5} + \cdots + \left(\frac{1}{n-4} - \cancel{\frac{1}{n-2}}\right) + \left(\frac{1}{n-3} - \cancel{\frac{1}{n-1}}\right) + \left(\cancel{\frac{1}{n-2}} - \boxed{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n-1}} - \boxed{\frac{1}{n+1}}\right)\Bigg)$$

$$S_n = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right) \quad \forall n \geq 2$$

$\boxed{\lim_{n\to\infty} S_n = \frac{3}{4}}$ Thus $\boxed{\sum_{n=2}^{\infty} \frac{1}{n^2-1} = 3/4}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges!

adding pizza slices! more pizza there

$\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 7/4$

Look at the partial sums

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2} \leq 1 + \sum_{k=2}^{n} \frac{1}{k^2-1} = 1 + \frac{3}{4} - \frac{1}{n} - \frac{1}{n+1} < \frac{7}{4}$$

pizza-party

$\forall n \in \mathbb{N}$

Clearly $\{S_n\}$ is an increasing sequence.
Thus by MCT it converges.
Not so easy to prove that

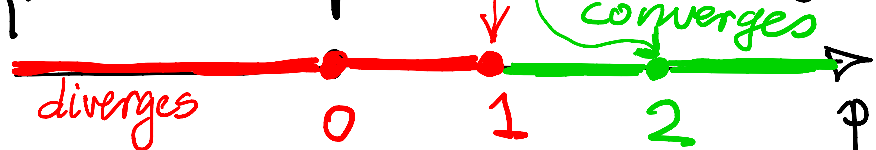
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$$

$p>0$ $$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

p-series

$p=1$ diverges

$p=2$ converges



The end of Special Infinite Series:
Geometric, Harmonic, Telescoping

We move to the General Theory of
CONVERGENT INFINITE SERIES

The most important theorem.

DIVERGENCE TEST: HUGE

Theorem: If $\sum_{n=1}^{\infty} a_n$ converges, then $\boxed{\lim_{n\to\infty} a_n = 0}$.

Proof. Assume $\boxed{\sum_{n=1}^{\infty} a_n \text{ converges.}}$ This means that the corresponding sequence of partial sums converges.

$\forall n \in \mathbb{N} \quad S_n = \sum_{k=1}^{n} a_k$ and

Say $\boxed{L = \lim_{n\to\infty} S_n}$

Yesterday I proved $\boxed{L = \lim_{n\to\infty} S_{n+1}}$

Algebra of convergent sequences:

If $A = \lim_{n\to\infty} u_n$ $B = \lim_{n\to\infty} v_n$ (convergent sequences)

$\{u_n - v_n\}$ new sequence is convergent and

$$\lim_{n\to\infty} (u_n - v_n) = A - B.$$

We know $L = \lim_{n\to\infty} S_n$, $L = \lim_{n\to\infty} S_{n+1}$

$$\lim_{n\to\infty} \underbrace{(S_{n+1} - S_n)}_{= a_n} = L - L = 0$$

thus

$$\boxed{\lim_{n\to\infty} a_n = 0}$$

THE CONVERSE is NOT TRUE

$$\lim_{n\to\infty} a_n = 0 \;\overset{\text{does not}}{\underset{\text{imply}}{\cancel{\Longrightarrow}}}\; \sum_{n=1}^{\infty} a_n \text{ converges}$$

The example is the Harmonic Series!

$$\boxed{\lim_{n\to\infty} \frac{1}{n} = 0 \text{ and } \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}}$$