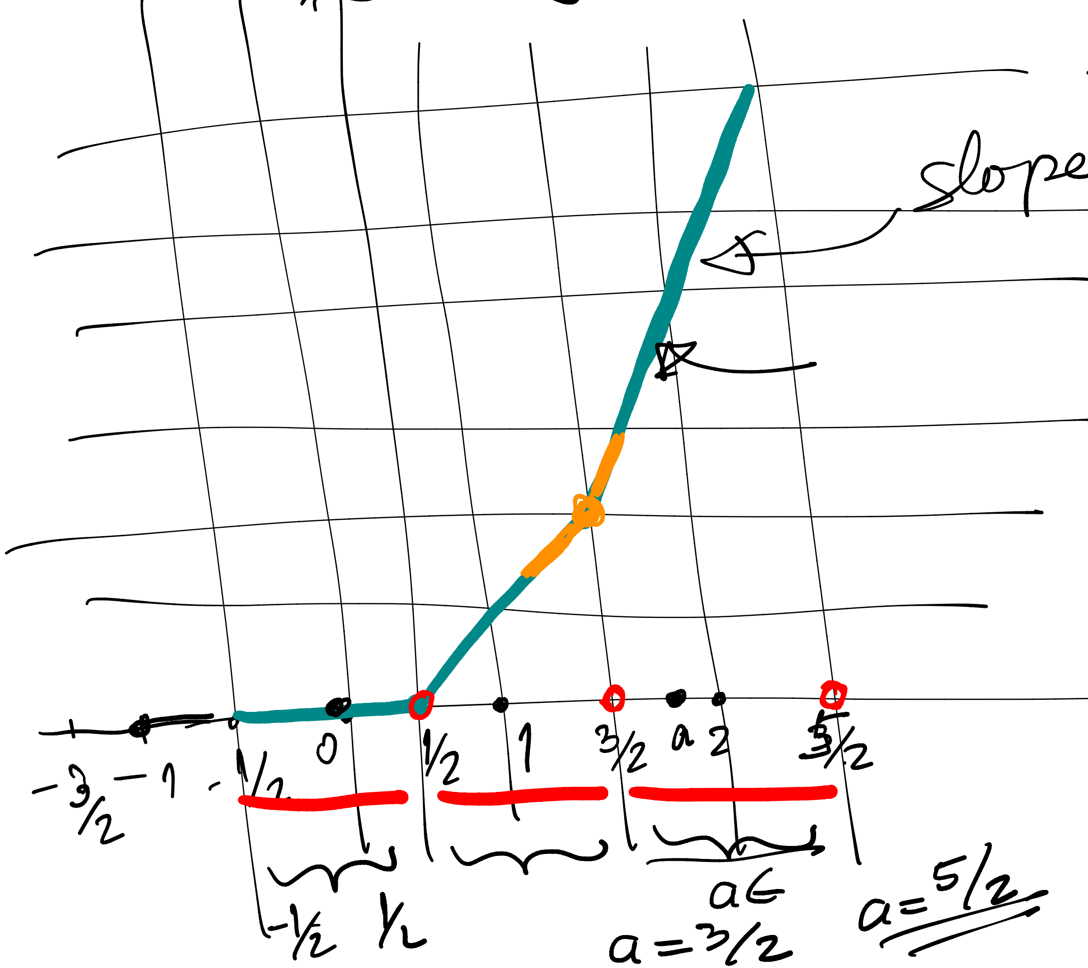
Comparison Tests

(also a hint for Problem 1 on A3)

June 1, 2020

Assignment 3 Problem 1



$$|f(x) - f(a)| < \varepsilon$$

Cases

Continuity at a

Case 1: $k \in \mathbb{Z}$

$$k - \frac{1}{2} < a < k + \frac{1}{2}$$

$$\delta(\varepsilon, a) = ---$$

Case 2

$$a = k + \frac{1}{2}$$

Study general Infinite Series:

$$\sum_{n=1}^{\infty} a_n$$

we want to determine whether it CONVERGES or DIVERGES

need TOOLS to determine

DIVERGENCE TEST ← the first tool!

We proved on Friday →

$$\boxed{\boxed{\sum_{n=1}^{\infty} a_n \text{ CONVERGES}} \Rightarrow \boxed{\lim_{n\to\infty} a_n = 0}}$$

CONTRAPOSITIVE: NOT

Not true $\boxed{\lim a_n = 0} \Rightarrow \sum a_n$ not diverges

this is a condition that assures divergence

$$\sum_{n=1}^{\infty} \frac{n^{(-1)^n}}{n+1}$$ Diverges

The sequence

$$\frac{n^{(-1)^n}}{n+1}$$

Does not converge !

$$\frac{n^{(-1)^n}}{n(n+1)} = \begin{cases} \frac{1}{n(n+1)} & n \text{ odd} \\ \frac{n}{n+1} & n \text{ even} \end{cases}$$

$$\lim_{n\to\infty} \frac{n^{(-1)^n}}{n+1} = 0$$ NOT true

The list of all tools:

- Divergence Test (universal)

- Comparison Test
- Limit Comparison test
- Integral Test

require $a_n > 0$ then

- Ratio Test
- Root test

use geometric series

Direct Comparison Test

$\boxed{0 < a_n \leq b_n}$ and $\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum a_n$ CONVERGES

$\sum b_n$ converges — friendly

We know this

$\sum_{n=1}^{\infty} a_n$????

Look at partial sums

$S_n^{(b)} = \sum_{k=1}^{n} b_k$ increasing converges

$S_n^{(a)} = \sum_{k=1}^{\infty} a_k \leq S_n^{(b)} \leq B$

increasing, bdd above, MCT $\Rightarrow$ Converges

$\boxed{S_n^{(b)} \leq \lim S_n^{(b)} = B}$

Example

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{1}{n^2} \le \frac{1}{n(n-1)}$$

$n \ge 2$

telescopic series

converges

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Math 430

$$\sum_{n=1}^{\infty} \frac{1}{n!} \quad \text{converges}$$

$$\frac{1}{n!} \le \frac{1}{2^{n-1}} \quad n \ge 2$$

geom. series.

Limit Convergence Test

Two series $\sum a_n$, $\sum b_n$

ONLY TWO ASSUMPTIONS

Assume

$$\boxed{\lim_{n\to\infty} \frac{a_n}{b_n} = L \in [0, +\infty)}$$

$\Downarrow$

Intuitive understanding:

$L > 0$ then $\boxed{a_n \approx L b_n}$

If $\boxed{\sum b_n \text{ converges}}$, then $\sum a_n$ conv.

$\sum b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} L b_n$ conv.

$L = 0$ $\boxed{\lim_{n\to\infty} \frac{a_n}{b_n} = 0}$ means (intuitively) b_n is much bigger than a_n $\Rightarrow \sum a_n$ conv.

Example

$$\underbrace{\sum_{n=1}^{\infty} \frac{n+1}{\sqrt{1+n^6}}}$$

$n+1 \sim n$

$\sqrt{n^6+1} \sim n^3$

$\frac{n}{n^3} \sim \frac{1}{n^2}$

$\sum \frac{1}{n^2} < \infty$ converges

$$\lim_{n\to\infty} \frac{\frac{n+1}{\sqrt{1+n^6}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3+n}{\sqrt{1+n^6}} = \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{\sqrt{1+\frac{1}{n^6}}} = 1$$