Integral Test

Ratio Test

June 2, 2020

Integral Test

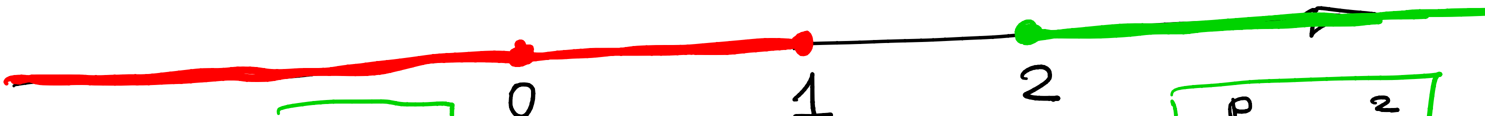
$$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad p \in \mathbb{R}$$

$p=1$ Harmonic Series

We did a rigorous proof that this series DIVER.

$p=2$ CONVERGES

rigorous proof.



$n \in \mathbb{N}$

$\boxed{p < 1} \Rightarrow \boxed{n^p < n}$

$p = 1/2$

$\boxed{\frac{1}{n^p} > \frac{1}{n}}$

Comparison Test,

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges $\Leftarrow \boxed{\sum \frac{1}{n} \text{ diverges}}$

$\boxed{p > 2} \Rightarrow \boxed{n^p > n^2}$

$\boxed{\frac{1}{n^p} < \frac{1}{n^2}}$

comp.

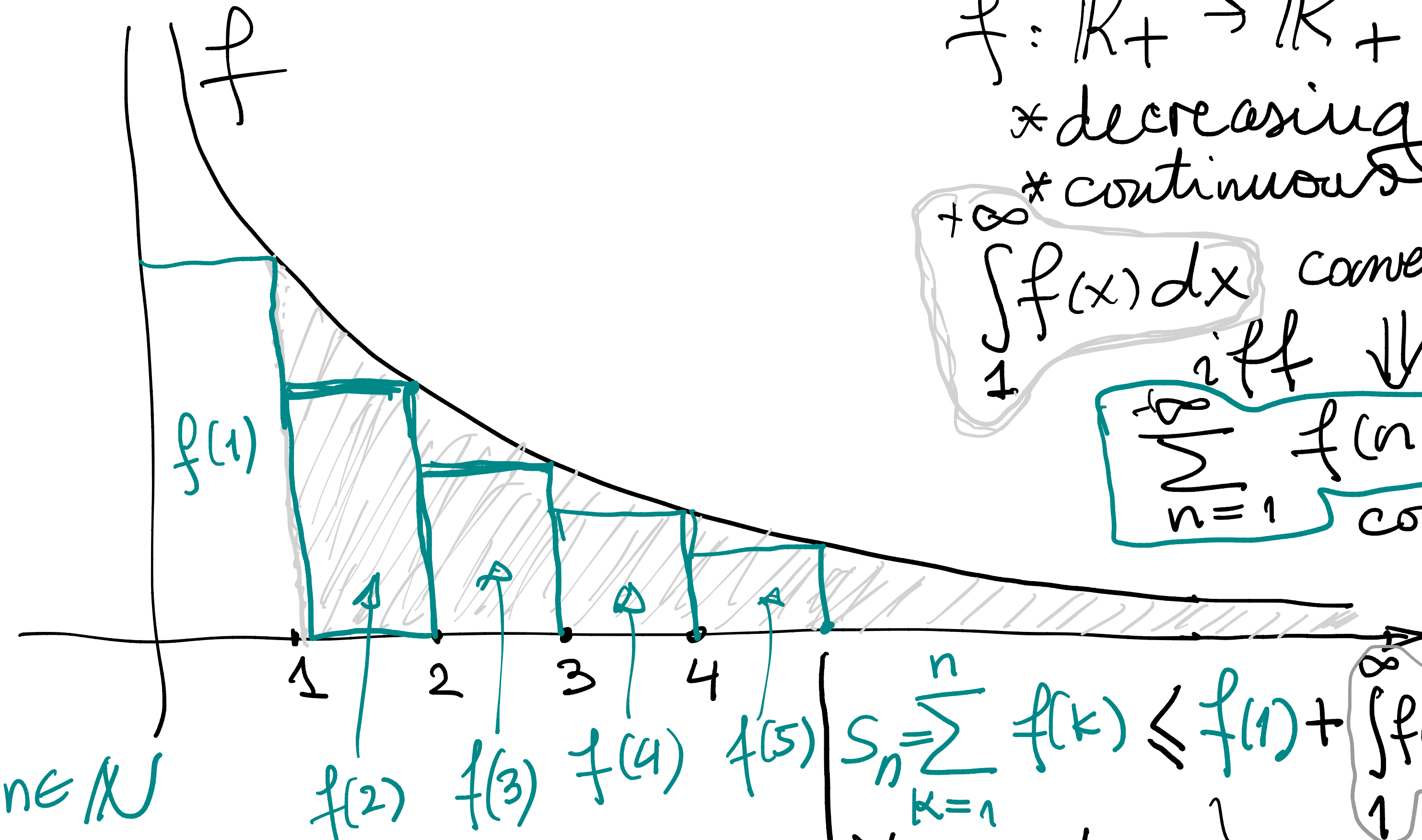
$\sum_{n=1}^{\infty} \frac{1}{n^p} \Leftarrow \sum \frac{1}{n^2}$

conv

For $p \in (1,2)$ I need the Integral Test.

$f : \mathbb{R}_+ \to \mathbb{R}_+$

* decreasing
* continuous



$\boxed{\int_1^{+\infty} f(x)\,dx}$ converges $\overset{2}{\text{iff}}$ $\boxed{\sum_{n=1}^{+\infty} f(n)}$ converges

$\Downarrow$ need this direction

$$S_n = \sum_{k=1}^{n} f(k) \leq \underbrace{f(1) + \int_1^{\infty} f(x)\,dx}_{M}$$

$\forall n \in \mathbb{N}$

$\{S_n\}$ the sequence of partial sums is increasing and bdd above so, it converges.

$(x^a)' = a x^{a-1}$

$\boxed{p > 1}$

$$\int_1^{+\infty} \frac{1}{x^p}\,dx = \lim_{X \to +\infty} \int_1^X \frac{1}{x^p}\,dx =$$

improper integral

$$\int x^{-p}\,dx = \frac{1}{1-p} x^{-p+1} = \frac{1}{1-p} \frac{1}{x^{p-1}}$$

antiderivative

$$\overset{FTC}{=} \lim_{X \to +\infty} \left(\frac{1}{1-p} \frac{1}{x^{p-1}} \Big|_1^X \right)$$

$$= \lim_{X \to +\infty} \frac{1}{1-p} \left(\frac{1}{X^{p-1}} - 1 \right)$$

$p > 1$

$p - 1 > 0$

$$= \left(\frac{1}{1-p} \text{ constant, we can prove } \lim_{x\to+\infty}\left(\frac{1}{x^{p-1}} - 1\right) = -1 \right)$$

$$= \frac{1}{1-p}(-1) = \frac{1}{p-1} > 0$$

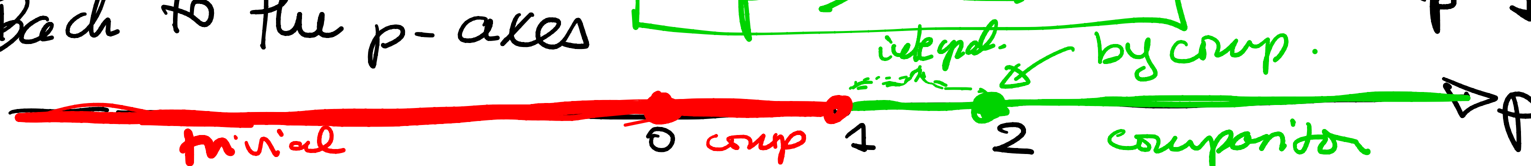
This shows that $\int_1^{+\infty} \frac{1}{x^p}\,dx = \frac{1}{p-1}$ (provided $p > 1$)

<u>CONVERGES</u>

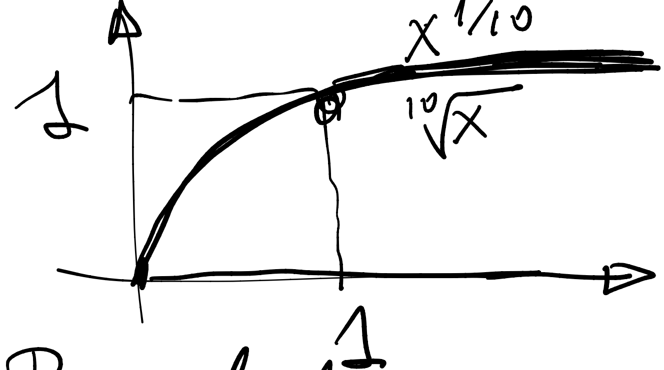
Therefore $\sum_{n=1}^{\infty} \frac{1}{n^p}$ <u>converges</u>, its sum $< 1 + \frac{1}{p-1} = \frac{p}{p-1}$

$$\boxed{p > 1}$$

Back to the p-axes



$$\underbrace{\sum \frac{1}{n}}_{\text{diverges}} \qquad \underbrace{\sum \frac{1}{n^2}}_{\text{converges}}$$



make n^1 bigger $\quad n^1 < n^{11/10} \qquad \sum \frac{1}{n^{11/10}}$

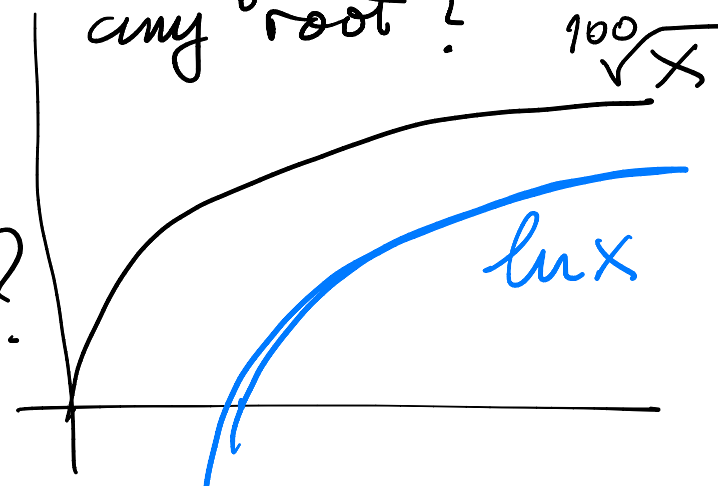
$$n < n \cdot n^{1/10}$$

$$n < n \cdot \ln n$$

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \quad ?$$

converges or diverges?

Precalculus: What grows slower than any root?



Amazing test which comes from geometric Series

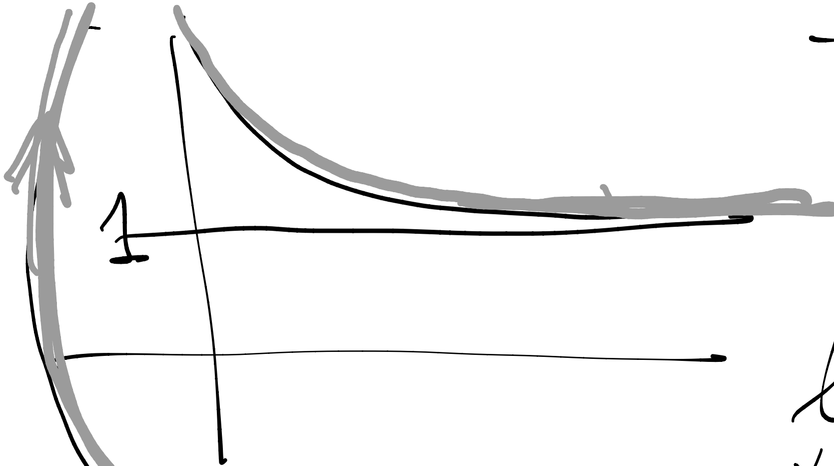
$$\sum_{n=1}^{\infty} a_n$$

Is this series a geometric series?

$$\boxed{\frac{a_{n+1}}{a_n} = r \quad \forall n \in \mathbb{N}}$$

Hard to achieve!

the exact equality



$$\boxed{\lim_{x\to+\infty} 1 = 1}$$

trivial

constant

$$\lim_{x\to+\infty} \frac{x+1}{x} = 1$$

The next best thing to this $\frac{a_{n+1}}{a_n}$ sequence being constant is

$$\boxed{\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = r}$$

We should have introduced a "slang": for function has a limit:

$\boxed{\text{Constantish}}$

$a_n > 0 \longrightarrow$ geometrisch series

$0 \leq r < 1 \Rightarrow$ converges

$r > 1 \Rightarrow$ diverges

RATIO TEST