Alternating Series
Conditional Convergence

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Alternating Series

The most prominent example is

$$\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

the Alternating Harmonic Series.

used in proof

In general: $\boxed{\forall n \in \mathbb{N} \quad a_n > 0}$ G1

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$ an Alternating Series

G2 $\boxed{\forall n \in \mathbb{N} \quad a_n \geq a_{n+1}}$ G3 $\boxed{\lim_{n\to+\infty} a_n = 0}$

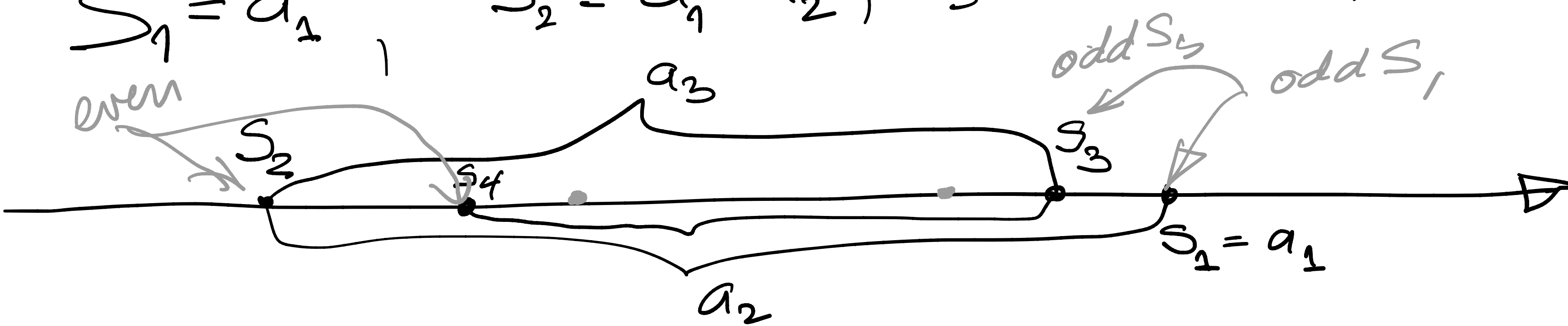
used in proof for $S_{2j} \leq S_{2k-1}$

If **G1**, **G2** and **G3** are satisfied, then the $\boxed{\text{Alternating Series converges}}$.

<u>Proof.</u> Study the partil sums:

$$S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k .$$

$S_1 = a_1$, $S_2 = a_1 - a_2$, $S_3 = S_2 + a_3$, $S_4 = S_3 - a_4$



A conjecture from the above picture is :

$$\boxed{\forall j \in \mathbb{N} \ \forall k \in \mathbb{N} \quad S_{2j} < S_{2k-1}}$$

Proved in the notes
$a_n \geq a_{n+1}$
$S_{2j} - S_{2k-1}$ must be < 0

This gives us a tool to create the limit of the sequence $\{S_n\}_{n=1}^{\infty}$ using

the Completeness Axiom

If $A, B \subseteq \mathbb{R}$, $A, B \neq \emptyset$,

$\boxed{\forall a \in A \ \forall b \in B \ \ a \le b} \Rightarrow \exists c \in \mathbb{R}$ s.t. $a \le c \le b$ $\forall a \in A \ \forall b \in B$

Completeness Axiom

no leaks

By CA, setting $A = \{ S_{2j} : j \in \mathbb{N} \}$
$B = \{ S_{2k-1} : k \in \mathbb{N} \}$,

$\exists c \in \mathbb{R}$ s.t. $\forall j \in \mathbb{N} \ \forall k \in \mathbb{N}$ $\boxed{S_{2j} \le c \le S_{2k-1}}$

Next thing to prove is $\lim_{n\to+\infty} S_n = c$

$\forall \varepsilon>0 \quad \exists N_S(\varepsilon)\in\mathbb{R}$ s.t.

$\forall n\in\mathbb{N} \quad n>N_S(\varepsilon) \Rightarrow |S_n - c| < \varepsilon$

How do I find $N_S(\varepsilon)$? How do I prove

Recall that $\boxed{\lim_{n\to+\infty} a_n = 0}$ and what this means is:

$\forall \varepsilon>0 \quad \exists N_a(\varepsilon)\in\mathbb{R}$ s.t.

$\forall n\in\mathbb{N} \quad n>N_a(\varepsilon) \Rightarrow \overbrace{|a_n - 0|<\varepsilon}^{a_n<\varepsilon}$

There must be a connection between $\boxed{}$ and $\boxed{}$

$$\boxed{\text{The connection is: } \left|S_{n+1} - S_n\right| = \left|(-1)^{n+2} a_{n+1}\right| \overset{\text{since } a_{n+1}>0}{=} a_{n+1}}$$

We know that <u>c is between</u> S_n and S_{n+1}

one is even
one is odd

Therefore $\forall n \in \mathbb{N}$

$$\boxed{\underbrace{\left|S_n - c\right|}_{\text{how can I make this} < \varepsilon} \leq \left|S_n - S_{n+1}\right| = a_{n+1}}$$

$\longrightarrow$ make this $< \varepsilon$

Let $\varepsilon > 0$ be arbitrary.

Set $N_s(\varepsilon) = N_a(\varepsilon)$

then $n \in \mathbb{N}$ $n > N_s(\varepsilon) \Rightarrow a_n < \varepsilon$

but $a_{n+1} \le a_n < \varepsilon$

and $\boxed{|S_n - c| \le a_{n+1}}$ implies $\boxed{|S_n - c| < \varepsilon}$

So, I greenified (which was red above)

$\forall n \in \mathbb{N}$ $n > N_s(\varepsilon) \boxed{\Rightarrow} |S_n - c| < \varepsilon$.

Thank You Carter. !!

This theorem is called Alternating Series Test

It follows from this Theorem that the Alternating Harmonic Series Converges

In the notes I prove that

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \ln 2$$

In fact $\forall p > 0$ $\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{1}{n^p}$ converges

There is something amazing about the Alt. Harm. Series.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$ (think of this is balancing a checkbook)

$$\boxed{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \ln 2}$$

$$D: 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2n-1}$$ deposits, series diverges $+\infty$

$$W: \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$$ withdrawals, series diverges $+\infty$

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \frac{1}{14} - \frac{1}{16} + \cdots$$

in the notes I show $= \frac{1}{2}\ln 2$

Conditional Convergence

$\sum_{n=1}^{\infty} b_n$ converges, but $\sum_{n=1}^{\infty} |b_n|$ diverges

we call $\sum b_n$ conditionally convergent

If both $\sum_{n=1}^{\infty} b_n$ converges and $\sum |b_n|$ conv.

Absolute convergence

$\sum (-1)^{n+1} \frac{1}{n^2}$ converge absolutely (unconditionally).