Functions: Floor, Ceiling, Round, & most importantly Absolute Value

A, B nonempty sets

$f : A \to B$

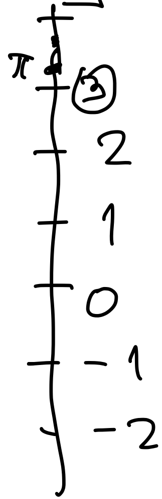
f associates exactly one elem. of B with <u>each</u> element of A.

the domain of f the <u>codomain</u> of f

the <u>range</u> of f is $\{f(x) \in B : x \in A\}$

Let us talk about the FLOOR funct.

$$\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \le x\}$$



looks down a lots of floors below me!

$$\text{floor} : \mathbb{R} \to \mathbb{R}$$

$\mathbb{Z}$

$\mathbb{Q}$ all rational

The range of floor is $\mathbb{Z}$.

Remember the max of a set must be in the set.

$\lfloor \pi \rfloor = 3$, $\lfloor e \rfloor = 2$, $\lfloor -\pi \rfloor = -4$, ...

Super important characterization of floor is

$m \in \mathbb{Z}$, $x \in \mathbb{R}$ $\quad m = \lfloor x \rfloor \iff \boxed{m \le x \wedge x < m+1}$ BK 1

$$\iff x-1 < m \wedge m \le x$$

In words: floor of x is the integer in $(x-1, x]$ (interval)

~~Exercise~~ Problem: Prove $\forall x \in \mathbb{R}$

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor.$$

Solution Consider two cases: BK1

By BK1 $0 \le x - m$ and $x - m < 1$. $m = \lfloor x \rfloor$

that is $x - m \in [0, 1)$

$$\boxed{x - \lfloor x \rfloor \in [0, 1)}$$

Case 1. $x - \lfloor x \rfloor \in [0, 1/2)$

Case 2. $x - \lfloor x \rfloor \in [1/2, 1)$

These two cases cover all $x \in \mathbb{R}$ since

union of two sets

$$\boxed{[0, 1) = [0, 1/2) \cup [1/2, 1)}$$

Case 1 $\quad 0 \leq x - \lfloor x \rfloor$ and $x - \lfloor x \rfloor < 1/2$

$\boxed{\lfloor x \rfloor \leq x}$ and $\boxed{x < \lfloor x \rfloor + 1/2}$

Red is $\quad \lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$

mult. by 2

$\underbrace{2\lfloor x \rfloor}_{\in \mathbb{Z}} \leq 2x$ and $2x < \underbrace{2\lfloor x \rfloor}_{\in \mathbb{Z}} + 1$

$\cancel{2x - 1 < 2\lfloor x \rfloor}$

Now we see that $\boxed{\underbrace{2\lfloor x \rfloor}_{\in \mathbb{Z}} \in (2x-1, 2x]}$

Thus $\quad 2\lfloor x \rfloor = \lfloor 2x \rfloor$

Recall: $x - \lfloor x \rfloor \in [0, 1/2) \Rightarrow \lfloor x + 1/2 \rfloor = \lfloor x \rfloor$

therefore $\lfloor 2x \rfloor = 2\lfloor x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$

Case 2. Do it!

Ceiling: $\lceil x \rceil = \min\{k \in \mathbb{Z} : k \geq x\}$

\lceil \rceil

google ceiling in LaTeX

$n \in \mathbb{Z}$ $x \in \mathbb{R}$

$n = \lceil x \rceil \iff x \leq n$ and $n - 1 < x$

$\iff x \leq n$ and $n < x + 1$

In words: $\lceil x \rceil$ is the integer in $[x, x+1)$

Round

Read about Round

The Absolute Value function

$x \in \mathbb{R}$

$$\mathrm{abs}(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

The most import. appl. of abs is the distance between real numbers.

abs: the absolute value function

What is the dist. between e and π, $\pi - e$

e^{π} $\qquad$ e^{π} and π^{e}

$$|e^{\pi} - \pi^{e}|$$

In general, for $a, x \in \mathbb{R}$ the distance between a & x is $|a-x|$.

<u>Thm</u> a, x and c are real and $c > 0$

$$\boxed{\underset{\text{Bell}}{|a-x|} \underset{\leq}{<} \underset{20\text{m}}{c} \overset{?}{\iff} x \in (a-c, a+c)}$$

Make a Story BBB Bellingham, Blaine, Burlington

20 m

Where am I if I am 20m from Bellingham?
(on I-5)

I am between Burlington and Blaine.

$$|\underset{\text{Bellingham}}{a} - \underset{\text{me}}{x}| < \underset{\text{20m}}{c} \iff x \in (a-c,\ a+c)$$

$\underbrace{|a-x|<c}_{\text{20m from Bellingham on I-5}}$

Burlington
20m South of Bell.

Blaine
20m North of Bell.

See one more page with the proof of $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$

Proof. Let $x \in \mathbb{R}$. We will prove that

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$$

Background Knowladge: BK1. $x - \lfloor x \rfloor \in [0,1)$

BK2. $y \in \mathbb{R}$, $m \in \mathbb{Z}$ $\quad m = \lfloor y \rfloor \Leftrightarrow m \leq y \wedge y < m+1$

BK3. $[0,1) = [0, 1/2) \cup [1/2, 1)$

union of two sets

Proof. Case 1. $x - \lfloor x \rfloor \in [0, 1/2)$. Then

$0 \leq x - \lfloor x \rfloor$ and $x - \lfloor x \rfloor < 1/2$. That is

$\lfloor x \rfloor \leq x$ and $x < \lfloor x \rfloor + 1/2$. Multiply both inequalities

by 2: $\quad 2\lfloor x \rfloor \leq 2x$ and $2x < 2\lfloor x \rfloor + 1$. By BK2.

$\lfloor 2x \rfloor = 2\lfloor x \rfloor$

Since $\boxed{\lfloor x\rfloor \le x \text{ and } x < \lfloor x\rfloor + 1/2}$ we have $\boxed{\lfloor x\rfloor + 1/2 \le x + 1/2 \text{ and } x + 1/2 < \lfloor x\rfloor + 1}$

Since $\boxed{\lfloor x\rfloor < \lfloor x\rfloor + 1/2}$ we have $\boxed{\lfloor x\rfloor \le x + 1/2 \text{ and } x + 1/2 < \lfloor x\rfloor + 1}$

By BK2 $\boxed{\lfloor x + 1/2\rfloor = \lfloor x\rfloor}$

We proved $\boxed{\lfloor 2x\rfloor = 2\lfloor x\rfloor}$ and $\boxed{\lfloor x + 1/2\rfloor = \lfloor x\rfloor}$

Therefore $\lfloor 2x\rfloor = \lfloor x\rfloor + \lfloor x\rfloor = \lfloor x\rfloor + \lfloor x + 1/2\rfloor$

<u>Case 2.</u> $\boxed{x - \lfloor x\rfloor \in [1/2, 1)}$

Then $\boxed{\lfloor x\rfloor + 1/2 \le x \text{ and } x < \lfloor x\rfloor + 1}$. Multiply by 2:

$\boxed{2\lfloor x\rfloor + 1 \leq 2x \text{ and } 2x < 2\lfloor x\rfloor + 2}$

By BK 2 we conclude that $\boxed{\lfloor 2x\rfloor = 2\lfloor x\rfloor + 1}$

Again we use and rewrite:

$\boxed{\lfloor x\rfloor + 1 \leq x + 1/2}$ and $\boxed{x + 1/2 < \lfloor x\rfloor + 3/2 < \lfloor x\rfloor + 2}$

By BK 2 we deduce $\boxed{\lfloor x + 1/2\rfloor = \lfloor x\rfloor + 1}$

$$\lfloor 2x\rfloor = 2\lfloor x\rfloor + 1 = \lfloor x\rfloor + \lfloor x\rfloor + 1 = \lfloor x\rfloor + \lfloor x + 1/2\rfloor$$

Thus we proved $\boxed{\lfloor 2x\rfloor = \lfloor x\rfloor + \lfloor x + 1/2\rfloor}$

Since two cases cover all possible cases the proof is complete.

Is this an easier proof?

Case 1 $\lfloor 2x \rfloor$ is even. Then $\boxed{\lfloor 2x \rfloor = 2k}$ $k \in \mathbb{Z}$

By BK2: $2k \leq 2x$ and $2x < 2k+1$. Divide by 2:

$\boxed{k \leq x \text{ and } x < k + \frac{1}{2} < k+1}$. Therefore $\boxed{\lfloor x \rfloor = k}$.

Since $\lfloor 2x \rfloor = 2k = 2\lfloor x \rfloor$, the proof is complete in this case.

Case 2. $\lfloor 2x \rfloor$ is odd. Then $\boxed{\lfloor 2x \rfloor = 2k+1 \text{ with } k \in \mathbb{Z}}$.

By BK2: $\boxed{2k+1 \leq 2x \text{ and } 2x < 2k+2}$ Divide by 2

$\boxed{k + \frac{1}{2} \leq x \text{ and } x < k+1}$ Then

$\boxed{k \leq x \text{ and } x < k+1}$ $\quad$ $\boxed{k+1 \leq x+\tfrac{1}{2} \text{ and } x+\tfrac{1}{2} < k+\tfrac{3}{2} < k+2}$

$\Downarrow$ By BK2 $\quad$ $\Downarrow$ By BK2

$\boxed{\lfloor x \rfloor = k}$ $\quad$ $\boxed{\lfloor x+\tfrac{1}{2} \rfloor = k+1}$

Now we summarize:

$$\lfloor 2x \rfloor = 2k+1 = k + k+1 = \lfloor x \rfloor + \lfloor x+\tfrac{1}{2} \rfloor .$$

Case 2