The Absolute Value function

and

its properties

$$|a+b| \leq |a| + |b|$$

and many others

The definition of the absolute value function is a piecewise definition:

$$\mathrm{abs}(x) = |x| = \begin{cases} -x & \text{if } x<0 \\ x & \text{if } x \geq 0 \end{cases}$$

notice two different notations; this is because computers prefer real names, not "symbols".

The most important properties of the absolute value function are given in the following two theorems.

<u>Theorem</u> (i) $\forall x \in \mathbb{R}$ $|x| = \max\{-x, x\}$.

This property is proved by considering two cases for $x \in \mathbb{R}$
Case 1 $x < 0$ and Case 2. $x \geq 0$. Since these are all possible cases, when we prove the property for each case the proof will be complete.

Case 1 $x<0$. By definition of abs $\boxed{|x| = -x}$.

Since $\boxed{x<0}$, by BK $\boxed{0<-x}$. By Axiom OT, we have $\boxed{x<-x}$. By definition of max, since $x<-x$ we have $\boxed{\max\{x,-x\} = -x}$

By these two boxes $\boxed{|x| = \max\{x,-x\}}$ in this case.

The proof of Case 2 is very similar.

(iii) $\forall x \in \mathbb{R} \quad |x| \geq 0$.

The proof considers two cases as the previous proof.

(iv) $\forall x \in \mathbb{R} \quad -x \leq |x|$ and $x \leq |x|$.

Proof. Recall that $\boxed{|x| = \max\{x,-x\}}$. By a property of max $\boxed{-x \leq \max\{x,-x\}}$ and $\boxed{x \leq \max\{x,-x\}}$

Therefore $-x \leq |x|$ and $x \leq |x|$

Again, I skipped property (iii) $\forall x \in \mathbb{R}$ $|-x| = |x|$

<u>Proof</u>. It is clear that the following two sets are equal

$$\boxed{\{x, -x\} = \{-x, -(-x)\}} \leftarrow \text{this is BK.}$$

Since $\boxed{-(-x) = x}$

Since the sets are equal they have the same maximum

$$\max\{x, -x\} = \max\{-x, -(-x)\}$$

$$\max\{x,-x\} = |x|, \quad \max\{-x,-(-x)\} = |-x|$$

These three green $=$ complete the proof.

(iv) $\forall x, y \in \mathbb{R}$ $|xy| = |x||y|$

<u>Case 1.</u> $\boxed{x \geq 0}$, $\boxed{y \geq 0}$. Then by BK $\boxed{xy \geq 0}$

By def. of abs $\boxed{|x| = x}$, $\boxed{|y| = y}$ and $\boxed{|xy| = xy}$ $= |x||y|$.

Case 2. $\boxed{x \geq 0}, \boxed{y < 0}$. Then by BK $xy \leq 0$.

By the definition of abs we have

$\boxed{|x| = x}$, $\boxed{|y| = -y}$, $\boxed{|xy| = -(xy)}$. thus

$$|xy| = -(xy) \underset{BK}{=} x(-y) = |x||y|.$$

Case 3. $x < 0, y \geq 0$ Case 4. $x < 0, y < 0$.

Proofs for these cases are similar.

(vi) $\forall x, y \in \mathbb{R}$ such that $y \neq 0$ we have $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$.

Proof. Notice that $y = 0 \iff |y| = 0$. Therefore

$$y \neq 0 \iff |y| \neq 0.$$

We first prove that $\left|\frac{1}{y}\right| = \frac{1}{|y|}$.

Case 1. $y > 0$. By BK (See Thm 1.1 (viii) on p. 6 of the class notes)

$\frac{1}{y} > 0$. Therefore $\left|\frac{1}{y}\right| = \frac{1}{y}$.

Also $|y| = y$.

$\Longrightarrow \left|\frac{1}{y}\right| = \frac{1}{|y|}$

Case 2. $\boxed{y < 0}$. By def of abs $\boxed{|y| = -y}$.

By BK $\boxed{\frac{1}{y} < 0}$. By def of abs $\boxed{\left|\frac{1}{y}\right| = -\frac{1}{y}}$.

Now we calculate

$$\left|\frac{1}{y}\right| = -\frac{1}{y} \underset{BK}{=} \frac{1}{-y} = \frac{1}{|y|}.$$

This completes the proof.

this part of the proof.

Now we prove $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$. We calculate

$$\left|\frac{x}{y}\right| \overset{BK}{=} \left|x \cdot \frac{1}{y}\right| \underset{\text{prop.}}{\overset{(v)}{=}} |x|\left|\frac{1}{y}\right| = |x|\frac{1}{|y|} \overset{BK}{=} \frac{|x|}{|y|}$$

Theorem TRIANGLE INEQUALITIES

(i) $\forall a, b \in \mathbb{R} \quad |a+b| \leq |a| + |b|$.

Proof: Let $a, b \in \mathbb{R}$. By the preceding Theorem we have

$\boxed{a \leq |a|}$ and $\boxed{-a \leq |a|}$

$\boxed{b \leq |b|}$ $\quad$ $\boxed{-b \leq |b|}$

Therefore $\boxed{(-a)+(-b) \leq |a|+|b|}$

Therefore $\boxed{a+b \leq |a|+|b|}$

By BK $\boxed{-(a+b) \leq |a|+|b|}$

We proved $\boxed{\begin{aligned} a+b &\le |a|+|b| \\ -(a+b) &\le |a|+|b| \end{aligned}}$

By definition of max we deduce

$$\boxed{\underbrace{\max\{a+b, -(a+b)\}}_{|a+b|\text{ preceding Thm}} \le |a|+|b|}$$

Complete the proof.

(22) $\forall x, y, z$ we have $\boxed{|x-y| \le |x-z| + |z-y|}$.

We use $\boxed{|a+b| \le |a|+|b|}$.

Set $a = x-z$, $b = z-y$.

$|a+b| = |\underbrace{(x-z)+(z-y)}_{=|x-y|}| \le |x-z| + |z-y|$

Hence

$\boxed{|x-y| \le |x-z| + |z-y|}$

(iii) $\forall x, y \in \mathbb{R}$ we have $\big||x|-|y|\big| \leq |x-y|$.

Again we use $\boxed{|a+b| \leq |a|+|b|}$.

Now we set $a = x-y$, $b = y$. Then

$$\boxed{|x| = |(x-y)+y| \leq |x-y|+|y|}$$

By BK $\boxed{|x|-|y| \leq |x-y|}$ ✓

Now we set $a = y-x$, $b = x$. Then

$$\boxed{|y| = |(y-x)+x| \leq |y-x|+|x|}$$

By BK $\boxed{|y|-|x| \leq |y-x|}$ ✓

We proved $\boxed{|x|-|y| \le |x-y|}$

$\boxed{|y|-|x| \le |y-x|}$

By BK $\boxed{|y|-|x| = -(|x|-|y|)}$

By the previous theorem $\boxed{|y-x| = |-(y-x)| = |x-y|}$

therefore this box becomes $\boxed{-(|x|-|y|) \le |x-y|}$

$\boxed{|x|-|y| \le |x-y|}$

copy

By definition of max: $\max\{|x|-|y|, -(|x|-|y|)\} \le |x-y|$

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$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\big||x|-|y|\big|$

These two complete the proof.